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AN ALTERNATIVE TO SET THEORY

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1. Introduction. Much of modern research into the foundations of mathematics is a continuation of nearly 250 years of research following upon the invention of the calculus. It has been directed to finding some coherent conceptual basis for the calculus in particular and analysis in general, a conceptual basis which would provide a foundation for analysis in the sense of providing a motivation to assist in suggesting new theorems and a criterion of rigor to assist in judging the proofs of theorems.

The set theory of Cantor, despite the difficulties inherent in its concepts, has been partly successful in providing such a conceptual basis. Further, because the concept of set is so elegant, convenient, powerful and intuitively compelling, set theory has become a universal mathematical language used in almost all of modern mathematics. Nevertheless, the difficulties inherent in set theory remain.

In this paper an alternative to set theory is discussed which provides not only a conceptual basis for analysis free from the difficulties that trouble set theory, but also a mathematical language with some technical advantages over set theory. The alternative to set theory is presented in terms of symbol theory: a language for expressing certain fundamental relationships between symbols just as set theory is a language for expressing relationships between sets. Like naive set theory, the conceptual scheme of symbol theory can motivate axiomatic theories which are adequate for the development of analysis. The purpose of this paper is not, however, to present such an axiomatic theory but rather to describe symbol theory and the conceptual picture of mathematics suggested by it. Incidental to the presentation will appear the advantages of symbol theory over set theory as a mathematical language.

Logical notation is used throughout the paper only as a convenient shorthand means for stating sentences. The variables 'x', 'y', etc., are to be thought of as ranging over the particular universe under discussion. The universal quantifiers '(x)', '(y)', etc., are to be understood as asserting that no matter what objects in the universe the variable denotes, the sentence following the quantifier is true. The existential quantifiers '(Ex)', '(Ey)', etc., assert that there is an object in the universe such that when the variable denotes that object, the sentence following the quantifier is true. The logical connective '$\supset$' expresses the conditional "if . . . then . . . ", the connective '$\equiv$' the equivalence " . . . if and only if . . . ", the connective '&' the conjunction " . . . and . . . ", the connective '$\vee$' the disjunction " . . . or . . . " and '$-$' expresses the negation.

2. Symbol theory. It is the intention in this section to introduce naive symbol theory in much the same manner that naive set theory is introduced to novices, namely, by discussion of simple examples.

Consider a universe of symbols (finite or infinite, it does not matter at the moment) which includes the following symbols:

(2.1) $$1, 2, 3, 4, \text{even}, \text{odd}, +, <.$$

The members of the universe are not to be considered as symbols without meaning but it is presumed that the correct use of the symbols is known (in some vague sense of "use" which will be clarified later). It is possible to recognize two relationships between members of the universe; or put in another way, two relations, which will be called the epsilon and the nu relations, have the universe of symbols as their common domain. Thus it is possible to recognize from the manner in which *any* two symbols of the universe are used whether or not one is epsilon related to the other and whether or not one is nu related to the other.

In order to express easily the epsilon and nu relationships between symbols of the universe the following simple language will be used. For any two symbols s and t of the universe "$s \epsilon t$" will be said to be true if s is epsilon related to t, and "$s \nu t$" will be said to be true if s is nu related to t. Further the convention will be maintained that an instance of a symbol from the universe is a name for that symbol. Thus, for example, when it is said that "$2 \epsilon \text{even}$" is true it can be concluded that the symbol '2' is epsilon related to the symbol 'even' and when it is said that "$-2 \nu \text{even}$" is true it can be concluded that '2' is not nu related to 'even'.

The two relations are incompatible. The epsilon relationship is expressed in English by sentences such as "two is even", while the nu relationship is expressed by sentences such as "two is not odd". Thus the epsilon relation between symbols is such that the following sentences are true:

$$2 \epsilon \text{even}, \quad 4 \epsilon \text{even}, \quad 1 \epsilon \text{odd}, \quad 3 \epsilon \text{odd};$$

and such that the epsilon relationships expressed by these sentences are the only ones holding between the symbols (2.1); that is, that all the following sentences, and more, are true:

$$-1 \epsilon \text{even}, \ -3 \epsilon \text{even}, \ -2 \epsilon \text{odd}, \ -4 \epsilon \text{odd}, \ -1 \epsilon 1, \ -1 \epsilon 2,$$
$$-\text{even} \epsilon 2, \ -\text{odd} \epsilon \text{even}, \ -+ \epsilon <, \ -< \epsilon +, \ -3 \epsilon <.$$

Of these latter sentences the first four are true for apparently different reasons than the others; one concludes that '1' is not epsilon related to 'even' for a different reason than one concludes that 'odd' is not epsilon related to 'even', or that '3' is not epsilon related to '<'. The difference can be more clearly understood in terms of the nu relation.

The nu relation between symbols is such that the following sentences are true:

$$2 \nu \text{odd}, \quad 4 \nu \text{odd}, \quad 1 \nu \text{even}, \quad 3 \nu \text{even}.$$

And again also that the following sentences are true:

$$-2\ \nu\ \text{even}, \qquad -4\ \nu\ \text{even}, \qquad -1\ \nu\ \text{odd}, \qquad -3\ \nu\ \text{odd}.$$

But, further, just as it can be concluded that 'odd' is not epsilon related to 'even' so can it also be concluded that 'odd' is not nu related to 'even', or that '3' is not nu related to '<' and so on. That is, it can be concluded that the only nu relationships holding between the symbols (2.1) are those indicated above and that therefore all of the following sentences, and more, are true:

$$-1\ \nu\ 1, \qquad -1\ \nu\ 2, \qquad -\text{even}\ \nu\ 2, \qquad -\text{odd}\ \nu\ \text{even},$$
$$-<\ \nu\ +, \qquad -3\ \nu\ <.$$

The symbols that are epsilon or nu related to a given predicate symbol form, so to speak, the domain of application of the given symbol.* Thus, for example, the symbols '1', '2', '3', '4', and other numerals form the domain of application of the predicate symbols 'even' and 'odd'. The symbols that are in the domain of application of a given symbol may have the symbol positively or negatively applied to them (are epsilon or respectively nu related to the given symbol). Some symbols such as '+' and '<' are not predicate symbols since there are no symbols in the universe of symbols epsilon or nu related to them. Although there are no symbols among those of (2.1) either epsilon or nu related to the symbols '1', '2', '3', and '4', in some enlarged universe of symbols there might be such symbols; for example, a symbol to which exactly two objects are epsilon related would be epsilon related to '2'.

It is important to distinguish between "$s\ \nu\ t$" and "$-s\ \epsilon\ t$", and between "$s\ \epsilon\ t$" and "$-s\ \nu\ t$". For any universe of symbols

$$(2.2) \qquad (x)(y) - [x\ \epsilon\ y\ \&\ x\ \nu\ y]$$

should certainly be true; symbols are never used in such a way that one is epsilon and nu related to another. Sentences logically equivalent with (2.2) are:

$$(x)(y)[x\ \epsilon\ y \supset - x\ \nu\ y] \quad \text{and} \quad (x)(y)[x\ \nu\ y \supset - x\ \epsilon\ y].$$

Hence from the truth of "$s\ \epsilon\ t$" can be concluded the truth of "$-s\ \nu\ t$", and from the truth of "$s\ \nu\ t$" can be concluded the truth of "$-s\ \epsilon\ t$". But the converse is not the case; the sentences

$$(x)(y)[-x\ \nu\ y \supset x\ \epsilon\ y] \quad \text{and} \quad (x)(y)[-x\ \epsilon\ y \supset x\ \nu\ y]$$

are not true since they are contradicted in the given universe of symbols, for example, when both x and y are '1', for '1' is neither epsilon or nu related to itself. Hence "$s\ \nu\ t$" is not equivalent to "$-s\ \epsilon\ t$", and "$s\ \epsilon\ t$" is not equivalent to "$-s\ \nu\ t$".

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* In [3] in place of 'ν' appeared 'm' and in place of (2.2) appeared: $(x)\ (y)\ [x\ \epsilon\ y \supset x\ m\ y]$. Thus the symbols that are m-related to a symbol form its domain of application. However "$s\ m\ t$" can be understood to be "$s\ \epsilon\ t \vee s\ \nu\ t$"; or conversely, "$s\ \nu\ t$" can be understood to be "$s\ m\ t\ \&\ -s\ \epsilon\ t$". In (4) '$\nu$' and '$\epsilon$' replaced '$m$' and '$\epsilon$'.

It can be said that "$-1\ \epsilon$ even" is true for a different reason than "$-3\ \epsilon <$" because the former sentence follows by (2.2) from "$1\ \nu$ even", while not only is "$-3\ \epsilon <$" true but so also is the sentence "$-3\ \nu <$".

Assume now that the universe of symbols being considered is enlarged so as to contain not only the symbols (2.1) but also all concatenations of members of the universe with one another; that is, consider now a universe of symbols containing the symbols (2.1) that is closed under concatenation. The enlarged universe will therefore contain, among others, the following symbols:

$$2 \text{ even}, \quad \text{odd } 3 \text{ even}, \quad 1<, \quad 1+1, \quad 2+1, \quad <1, \quad <2, \quad <3, \quad <4.$$

The domain of the epsilon and nu relations can be extended to the enlarged universe; between any two members of the universe the epsilon relationship either does or does not hold and the nu relationship either does or does not hold. Clearly some of the concatenations of symbols from (2.1) have nothing either epsilon or nu related to them; for example, no symbol is either epsilon or nu related to the symbol '2 even' or to the symbol 'odd 3 even'. But the following sentences are true:

$$1\ \epsilon < 2, \quad 1\ \epsilon < 3, \quad 2\ \epsilon < 3, \quad 1\ \epsilon < 4, \quad 2\ \epsilon < 4, \quad 3\ \epsilon < 4,$$

as well as the following sentences:

$$1\ \nu < 1, \quad 2\ \nu < 1, \quad 3\ \nu < 1, \quad 4\ \nu < 1, \quad 2\ \nu < 2, \quad 3\ \nu < 2, \quad 4\ \nu < 2, \text{ etc.}$$

Thus, although nothing is epsilon or nu related to the symbol '$<$', there are symbols epsilon and nu related to the concatenation of '$<$' with numerals; such a symbol is called a (two place) relation symbol.

Relation symbols of more than two places are also possible. The most elegant manner of considering them is to presume that instead of the universe being closed under concatenation it is closed rather under the simultaneous concatenation of two symbols and the enclosing of the concatenate in brackets; consequently, that for any two symbols s and t in the universe, the symbol (st), the concatenation of left bracket, s, t and right bracket in that order, is a member of the universe. Then '$<$' is a relation symbol of two places because there are symbols epsilon or nu related to, for example, the symbol '(<2)'. A relation symbol of three places is a symbol 'R', for example, for which there are symbols epsilon or nu related to, for example, the symbol '$((R1)2)$'. Similarly relation symbols of more than three places can be introduced. However, for the convenience of avoiding the writing of extra brackets, instances of symbols like '<1' can be assumed to denote '(<1)', and instances of symbols like '$1R2$' or '$(1R2)$' can be assumed to denote the symbol '$((R1)2)$'.

3. Symbol theory and conventionalism. Assuming that the epsilon and nu relations are relations between symbols which are available for the specification of the usages of symbols, then a plausible conventionalist view of mathematics can be described. It can be assumed of each primitive predicate symbol appearing in a universe of symbols that it has been added to the universe and its use

for that universe specified by a statement describing what objects are to be epsilon related to it and what objects are to be nu related to it. Thus, for example, a predicate symbol 'L' could be added to a universe of symbols by specifying that it is to be used in the universe in such a way as to satisfy:

(3.1) $(x)\{[x \epsilon L \equiv (u) - u \epsilon x \,\&\, (Eu)u \nu x] \,\&\, [x \nu L \equiv (Eu)u \epsilon x]\}$.*

'L' is a symbol to which those symbols are to be epsilon or nu related which themselves have something epsilon or nu related to them; if objects are only nu related to a symbol then that symbol is epsilon related to 'L', and if objects are epsilon related to a symbol then that symbol is nu related to 'L'; 'L' is thus a symbol corresponding to the empty set of set theory. Since 'L' has been added to the universe of symbols, the universal quantifiers in (3.1) must be assumed to include 'L' within its range; but (3.1) is the only specification needed for the use of 'L'. Thus, for example, assuming that there is another symbol appearing in the universe of symbols that is epsilon related to 'L' then from (3.1) can be concluded that 'L' is nu related to itself.

When a symbol is added to the universe of symbols it must be assumed that other symbols are also added so as to close the new universe with respect to the simultaneous concatenation of two symbols and enclosing in brackets. It can then be assumed of each primitive relation symbol appearing in a universe of symbols that it has been added to the universe and its use for that universe specified by a statement describing what objects are epsilon or nu related to the concatenation of it with other symbols. For example, the usage of a three place relation symbol 'U' can be specified by:

(3.2) $(x)(y)(u)\{[u \epsilon (xUy) \equiv u \epsilon x \vee u \epsilon y] \,\&\, [u \nu (xUy) \equiv u \nu x \,\&\, u \nu y]\}$.

Here it is not specified what symbols are epsilon and nu related to 'U' (although a sentence similar in form to (3.1) could specify that no symbols are epsilon or nu related to 'U') but rather it is specified what symbols are epsilon and nu related to complex symbols involving 'U'. For example, (3.2) specifies that those symbols are to be epsilon related to '(LUL)' which are epsilon related to 'L', and specifies that those symbols are to be nu related to '$(LU(LUL))$' which are nu related to 'L' and nu related to '(LUL)'.

The statements which specify the usages of symbols may, actually, have to assert more about the usage of the symbols than (3.1) does for 'L' or than (3.2) does for 'U', for no mention has been made yet of the important distinction between extensional and intensional symbols or of the definition of identity.

The sentence

(3.3) $2 = (1 + 1)$

cannot be understood in the usual sense of identity as asserting that '2' denotes

* Here and below the logical connectives '&' and '$\vee$' are presumed to bind more strongly than the connectives '$\supset$' and '$\equiv$'. This means here, for example, that the sentence is to be read as though '$(u) - u \epsilon x \,\&\, (Eu)\, u \nu x$' were enclosed in brackets.

the same object as '(1+1)'. For, from the convention that has been adopted that an instance of a symbol of the universe is a name for that symbol, one would conclude that '2' and '(1+1)' were the same symbols while they are clearly not. (3.3) can be satisfactorily interpreted, however, if identity is understood as extensional identity; where x and y are extensionally identical (predicate symbols), in symbols $x=y$, if and only if

$$(3.4) \qquad (u)\{[u \in x \equiv u \in y] \,\&\, [u \nu x \equiv u \nu y]\}$$

is true. Hence (3.3) is understood as asserting that to '2' and to '(1+1)' exactly the same things are epsilon and nu related.

It is evident, however, that extensional identity does not have all the properties of identity. In particular it is not the case that for any z, the following is true:

$$(3.5) \qquad (x)(y)(x = y \supset (x \in z \equiv y \in z) \,\&\, (x \nu z \equiv y \nu z));$$

for example, should z be the symbol 'A-shaped' then it is possible for two symbols to be extensionally identical although only one is epsilon related to 'A-shaped'. In set theory a sentence similar to (3.5) is accepted as an axiom, the axiom of extensionality. But since the acceptance of (3.5) as an axiom in symbol theory would exclude the possibility of introducing intensional predicate symbols, (3.5) can better be seen as a definition of "z is extensional", and the assumption that a symbol is extensional be seen as a possible additional assumption to be made of its use. A symbol 'L' whose usage is specified by (3.1) can be shown to be extensional, but a symbol 'B' whose usage is specified by

$$(3.6) \qquad (x)\{[x \in B \equiv (Eu)x \in u] \,\&\, [x \nu B \equiv (u)x \nu u)\}$$

cannot be shown to be extensional; for "$(Eu)\ x \in u$" could be true because x is epsilon related to 'A-shaped' and therefore "$(Eu)\ y \in u$" could be false even though x and y are extensionally identical. If it is necessary that there be an extensional symbol 'B' satisfying (3.6), then 'B' must be assumed to be extensional; that is, 'B' is not only assumed to satisfy (3.6) but also to satisfy (3.5) with 'z' replaced by 'B'. With the additional assumption of extensionality it is possible to conclude in particular

$$[2 \in B \equiv (1 + 1) \in B] \,\&\, [2 \nu B \equiv (1 + 1) \nu B]$$

from (3.3); extensional identity has, therefore, all the properties required of identity with respect to extensional predicate symbols.

Extensional identity for relation symbols of any number of places can be defined similarly to extensional identity for predicate symbols. For example, extensional identity of two place relation symbols x and y is defined by:

$$(3.7) \qquad (u)(v)\{[u \in (xv) \equiv u \in (yv)] \,\&\, [u \nu (xv) \equiv u \nu (yv)]\},$$

and of three place relation symbols x and y is defined by:

(3.8) $(u)(v)(w)\{[u \epsilon ((xv)w) \equiv u \epsilon ((yv)w)] \,\&\, [u \nu ((xv)w) \equiv u \nu ((yv)w)]\}$.

For any given number, the extensionality of predicate symbols with respect to the extensional identity of relation symbols of that number of places can be defined by (3.5) when the identity in (3.5) is understood as the identity of relation symbols of the given number of places. But it is not possible to give a single definition of extensional identity for relations of any number of places, nor is it possible to give a single definition for the extensionality of predicate symbols with respect to the identity of relation symbols of any number of places. Therefore it is a great convenience, although not a necessity, to be able to assume that only relation symbols of up to three places occur as "primitive" and that relation symbols of more than three places can be constructed from them. For then a general definition of extensional identity can be given: x and y are extensionally identical if and only if they satisfy the conjunction of (3.4), (3.7) and (3.8). Further a definition of "z is an extensional predicate or relation symbol" can be given which is the conjunction of (3.5) and other sentences asserting that anything in either of the two places of a two place relation symbol, or in one of the three places of a three place relation symbol, can be replaced by anything else which is extensionally identical with it. With a general definition of extensional identity it is possible to have relation symbols of more than three places by compounding relation symbols of up to three places. It is only necessary to introduce a relation symbol 'P' of three places, the ordered pair relation symbol, in much the same manner that the ordered pair is defined in set theory. From the statement specifying the use of 'P' one should be able to conclude:

$$(x)(y)(u)(v)[(xPy) = (uPv) \supset x = u \,\&\, y = v].$$

Then a symbol y satisfying, for example,

$$(Eu)(Ev)(Ew)(Ex)[u \epsilon (vy)(wPx)) \vee u \nu (vy)(wPx))]$$

is a relation symbol of four places.

Symbols are ideal oriented shapes which do not of themselves bear epsilon or nu relationships with symbols or any other objects. Only when a symbol has been selected for membership in a universe of symbols (and there are no *a priori* dictates as to which symbols should be selected) and a statement specifying its usage has been given can anything be said to be epsilon or nu related to the symbol or can it be said to be epsilon or nu related to any other symbol or itself. The uses that are to be specified for the symbols selected will depend upon the purpose to which the universe of symbols is to be put. Certainly it is not to be expected that some particular universe of symbols together with the sentences specifying the usage of the symbols would, for some intrinsic reason, be *the* universe. Thus, for example, although it is possible to show that the simple theory of types can be understood to be a universe of symbols with their usages specified by sentences similar to (3.1) and (3.2) with an additional assumption of

extensionality,* it cannot be concluded that the universe of the simple theory of types is in any sense the correct one, or that the universe cannot be extended. In brief, mathematical truth is seen as truth relative to the particular universe of symbols with its statements of usage under consideration and not as truth independent of all universes of symbols.

4. Set theory. Unless one is only interested in formal systems or axiomatic theories and looks upon these systems as the mathematician's only legitimate interest, one must be interested also in the conceptual scheme which motivates the axiomatic set theories. Indeed, the concept of set—a collection of definite, distinct objects of our intuition or of our intellect, to give the best known "definition" from [2],†—and the relationship of membership between members of a set and the set, are what are central to set theory, not any particular axiomatic set theory. Axiomatic set theories are but attempts to describe a universe of sets; they are motivated by the conceptual scheme of naive set theory and derive any importance that they may possess from the conceptual scheme that motivates them.

As indicated in (6), for example, the manner in which sets are regarded varies from an extreme platonism to a conceptualism. From the platonist view sets are abstract objects existing independently of man, unchanging in some ideal world. From the conceptualist view sets are objects which are somehow constructed by man and maintain an existence after being constructed. But as Weyl shows in (8) with regard to the particular conceptualism that he espoused, the acceptance of the view leads to a weakening of analysis.

The conceptual basis provided for mathematics by the platonic view of set theory is unrivaled in elegance and simplicity. The truths of mathematics are discovered through the examination of the ideal world of sets; a theorem of analysis, for example, is a factual statement describing a state of affairs in the world. But despite the elegance, simplicity and power of this naive set theory it must be discarded as a foundation for mathematics because of the inconsistency of its concepts. The paradoxical sets of set theory can of course be excluded by accepting any one of a variety of ad hoc restrictions on the concept of set which have been investigated. Yet, no matter how the restrictions are formulated, the generality of the concept of set permits one to legitimately ask "but why does

---

* More precisely it can be shown that the simple theory of types without an axiom of infinity is equivalent with a theory S formalized within the first order predicate calculus. The theory S has ϵ and ν as its only primitive relations, (2.2) as its only axiom and an axiom scheme each instance of which is a sentence specifying the usage of a single symbol. Details will be given in my forthcoming paper *An interpretation of the simple theory of types*. A similar result was stated without proof in [3]. The result is stated in more detail and in a slightly different form in [4]; an interpretation is provided in [3], however, for only the monadic simple theory of types rather than the full simple theory of types.

† It is a translation of "Unter einer 'Menge' verstehen wir jede Zusammenfassung M von bestimmten wohlunterschiedenen Objecten m unserer Anschauung oder unseres Denkens (welche die 'Elemente' von M genannt werden) an einem Ganzen," taken from [1].

not one of the forbidden sets exist?" The question is not answered by pointing out that otherwise an inconsistency can be obtained, for that can only be an answer to the question "Why cannot this sentence be added as an axiom to this theory?" The question can be better worded "If a set is a collection of definite distinct objects of our intuition or of our intellect conceived of as a whole, then why cannot this particular collection of definite distinct objects be conceived of as a whole?" The paradoxes of set theory *are* paradoxical and are the main objection to using set theory as a conceptual basis for mathematics.

5. Symbol theory as an alternative to set theory. To the view of mathematics suggested by set theory there exists many alternatives: intuitionism and formalism in all their various forms are two examples and the conventionalist view described in Section 3 is another. But symbol theory is also an alternative to set theory in the immediate sense of being able to replace set theory in the many uses made of the latter in mathematics. However, as the paradoxes of set theory are considered to be the main obstacle to the acceptance of set theory as a conceptual basis for mathematics, it is necessary to examine the paradoxes from the point of view of symbol theory.

The epsilon and nu relationships that hold between symbols of any universe of symbols that has been selected, hold only by virtue of the specified conventions of usage and not by virtue of any intrinsic properties of the symbols. Thus, for example, that the symbol 'L' whose use is specified by (3.1) will be epsilon related to itself in some universes of symbols is a fact which is a consequence of the convention (3.1) and conventions for other symbols of the universes and does not follow from the nature of the symbol 'L' itself.

It is of course possible to specify conventions that cannot possibly be satisfied; for example, it is possible to specify that 'A' be used as indicated by:

$$(5.1) \qquad (x)\{[x \epsilon A \equiv x = x] \,\&\, [x \nu A \equiv x = x]\}.$$

But from (2.2) it is impossible that any symbol satisfy (5.1)—it can only be through a misunderstanding that one would insist that 'A' do so. So also it is possible to insist that the use of the symbol 'R' is specified by:

$$(5.2) \qquad (x)\{[x \epsilon R \equiv - x \epsilon x] \,\&\, [x \nu R \equiv x \epsilon x]\}$$

although one would do so in ignorance, for by substituting 'R' for x,

$$'[R \epsilon R \equiv - R \epsilon R]'$$

can be concluded from (5.2) and therefore "$-R \epsilon R \,\&\, R \epsilon R$" is a consequence of (5.2). It is therefore not paradoxical that the symbol 'R', nor any other symbol, cannot satisfy (5.2) since there is a simple explanation of why it cannot do so: namely, the sentence (5.2) is contradictory.

Although it is not possible for any symbol to satisfy (5.2) it is possible for there to exist a predicate symbol with properties analogous to the properties of Russell's paradoxical set of all sets which are not members of themselves. For

the usage of such a predicate symbol 'R' is specified by:

$$(5.3) \qquad (x)\{[x \in R \equiv x \nu x] \,\&\, [x \nu R \equiv x \in x]\}.$$

This specification of usage is not contradictory nor does it contradict (2.2). By substituting 'R' for x, "$[R \in R \equiv R \nu R]$" can be concluded from (5.3), and therefore "$-R \in R \,\&\, -R \nu R$" can be concluded from (5.3) and (2.2). Thus it is possible for there to be a Russell predicate symbol, although one must conclude that such a symbol is neither epsilon nor nu related to itself.

The fact that the usage of a Russell symbol 'R' can be consistently specified by (5.3) although not by (5.2), and that the most important difference between these two statements is in the replacement of '$-x \in x$' by '$x \nu x$' might suggest that symbol theory provides an explanation of the Russell (and other) paradoxes in the distinction between "$-s \in t$" and "$s \nu t$". The validity of the conclusion depends upon the sense of the word "explanation" intended. It cannot be denied that the replacement of '$-x \in x$' by '$x \nu x$' in (5.2) converts it into a satisfiable sentence; but that is not to say that the paradoxes are avoided by introducing the nu relation in addition to the epsilon relation. The paradoxes would not be avoided in set theory by recognizing in some way the nu relationship as well as the epsilon relationship between sets; for the paradoxical set of sets that are not epsilon related to themselves would persist in plaguing such a theory and would be joined by the paradoxical set of sets that are not nu related to themselves. Symbol theory avoids the difficulties of the paradoxes not by virtue of the nu relation, although the possibility of consistently introducing a Russell predicate symbol argues for symbol theory; rather the difficulties are avoided by recognizing that the epsilon and nu relationships between the basic objects of mathematics are not intrinsic to the objects but are ones that arise only from assumptions of conventional usage, and that therefore contradictory assumptions of usage are the cause of contradictory relationships between the objects. If by "explanation of the paradoxes" is meant, however, insight into how to avoid specifying usages for symbols that cannot be fulfilled, then the distinction between "$-s \in t$" and "$s \nu t$" is a partial explanation of the paradoxes; but it is, of course, only partial.

An objection to symbol theory, however, might be raised. It might be argued that if symbol theory is able to replace set theory as a mathematical language then one is led to a contradiction which provides an argument against symbol theory as a foundation for mathematics. For by Cantor's diagonal method of proof it can be shown that the universe of sets is nondenumerable. If symbol theory is able to replace set theory then it must be possible to duplicate this proof in some specified symbol theory and show that the universe of symbols for that theory is nondenumerable. But clearly at most denumerably many primitive symbols can be added to the universe and the universe of all concatenations of these symbols will also be denumerable. Here, apparently, is a contradiction of a different kind than the paradoxes of set theory, but one which nevertheless

makes symbol theory unsuitable as a foundation for mathematics, just as the paradoxes make set theory unsuitable.

A careful examination of the above argument can turn it equally well against set theory. From the point of view of set theory denumerability is a property of sets which is intrinsic to sets since there either does or does not exist in the universe of sets a one-one correspondence between a given set and the set of positive integers. From the point of view of symbol theory, on the other hand, 'denumerable' is a symbol which can be added to the universe of symbols by specifying a convention of usage. The convention will specify that a given symbol be epsilon related to 'denumerable' if and only if there exists in the universe of symbols a symbol which is a one-one correspondence between the symbols epsilon related to the given symbol and the symbols epsilon related to 'positive integer' (assuming some commonly accepted usage of this symbol has been given). But there will exist such a correspondence in the universe of symbols if and only if one has been added; thus although a given symbol may not be epsilon related to 'denumerable' in some universe of symbols, it is possible that the universe can be enlarged by the addition of new symbols so that in the enlarged universe the given symbol is epsilon related to 'denumerable.' From the point of view of set theory this relativism that is inherent in symbol theory is a fundamental flaw which argues against the acceptance of symbol theory as a conceptual basis for mathematics. But equally well from the point of view of symbol theory the absolutism inherent in set theory is a fundamental flaw. Neither theory is, on this point, of itself a sufficient argument for discarding the other.

The contrast between the two points of view is reminiscent of what is generally referred to as Skolem's paradox, discussed in detail in [5]. In [7], Skolem extended a theorem of Lowenheim to prove that no matter what intuitive conceptual scheme motivates a formal theory, an interpretation or model for the formal theory can be obtained with a denumerable universe of discourse. Thus in particular for any formal set theory whatsoever there can be found a denumerable model. It is possible to conclude from Skolem's result that no formal set theory is adequate for the description of the reality of the world of sets and that in particular a formal description of nondenumerable universes is impossible. However in view of the extraordinary richness of the formal theories that are available to the mathematician and in view of the contradictions inherent in set theory, it might be more reasonable to conclude that denumerability and nondenumerability are concepts that are more subtle than is suggested by naive set theory. Indeed Skolem suggested this conclusion and proposed a relativization of the concepts which, taking into account that he spoke of set theory, is very similar to that just outlined.

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QUADRICS IN R_n

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Techniques of vector and matrix algebra have been used in geometry of R_n, a real unitary n-dimensional space. We shall combine these techniques with use of certain homogeneous systems to treat properties of a quadric, *i.e.*, a complete equation of second degree in n variables. Then the results developed can be applied to the cases of two and three dimensions which actually cut down considerably the arithmetic of many problems.

We assume that the reader is familiar with elementary ideas of linear spaces, matrices, proper values and directions of a matrix, and reduction of quadratic forms. We denote any square matrix by a capital letter. The determinant of a matrix A is denoted by $\det A$; the inner product of two vectors ξ and η is denoted by (ξ, η).

1. Straight line. We begin with a short review of the vector and matrix equation of a straight line in R_n. Let $\eta \in R_n$ be fixed, and end on a straight line d. Let δ be a direction for d, *i.e.*, difference of two vectors ending on d. Then $\xi = \eta + t\delta$ is the equation of the line d. Clearly we can write this equation in matrix form such as

$$\begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix} = \begin{pmatrix} a_1 \\ \cdot \\ \cdot \\ \cdot \\ a_n \end{pmatrix} + t \begin{pmatrix} l_1 \\ \cdot \\ \cdot \\ \cdot \\ l_n \end{pmatrix},$$

where $\xi = (x_1, \cdots, x_n)$, $\eta = (a_1, \cdots, a_n)$, and $\delta = (l_1, \cdots, l_n)$, and t is a parameter. But for reasons to be seen later we write this equation as

$$\begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \\ 1 \end{pmatrix} = \begin{pmatrix} a_1 \\ \cdot \\ \cdot \\ \cdot \\ a_n \\ 1 \end{pmatrix} + t \begin{pmatrix} l_1 \\ \cdot \\ \cdot \\ \cdot \\ l_n \\ 0 \end{pmatrix},$$

or in row matrices. We note that to each point, or vector ξ in R_n corresponds a matrix $(x_1 \cdots x_n)$ or $(x_1 \cdots x_n\, 1)$ and to each direction δ corresponds a matrix $(l_1 \cdots l_n)$ or $(l_1 \cdots l_n 0)$. For convenience we denote these matrices by ξ and δ, and we use whichever form is suitable.

An equation of the first degree in $x_1, \cdots, x_n$ is called a plane in R_n, $n \geqq 3$.

2. Quadric. Let $\xi=(x_1 \cdots x_n\ 1)$ and A be a symmetric $(n+1)\times(n+1)$ matrix. Then a quadric can be written as $(A\xi, \xi)=0$. For example in R_3,

$$ax^2 + by^2 + cz^2 + 2hxy + 2gxz + 2fyz + 2px + 2qy + 2rz + d = 0$$

can be written as

$$(x \quad y \quad z \quad 1) \begin{pmatrix} a & h & g & p \\ h & b & f & q \\ g & f & c & r \\ p & q & r & d \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = 0.$$

Of course the use of these particular homogeneous coordinates $(x_1, \cdots, x_n, 1)$ made it possible to express a quadric as $(A\xi, \xi)=0$. For later uses we call the matrix of the quadratic part of the quadric Q.

3. Intersection of a straight line and a quadric. Let $(A\xi, \xi)=0$ be a quadric and $\xi=\eta+t\delta$ be a straight line in R_n. Then the points of intersection are solutions of the system

$$(A\xi, \xi) = 0, \qquad \xi = \eta + t\delta.$$

Eliminating ξ we get

$$(A\delta, \delta)t^2 + 2(A\eta, \delta)t + (A\eta, \eta) = 0. \tag{3.1}$$

Here since R_n is a real space and A is symmetric $(A\delta, \eta)$ has been replaced by $(A\eta, \delta)$. We also observe that $(A\delta, \delta) = (Q\delta, \delta)$. Thus (3.1) is written as

$$(Q\delta, \delta)t^2 + 2(A\eta, \delta)t + (A\eta, \eta) = 0. \tag{3.2}$$

Clearly the following cases can be considered:

(a) If $(Q\delta, \delta)\neq 0$, the line intersects the quadric in two points, real or complex conjugate or coincident.

(b) If $(Q\delta, \delta)=0$, $(A\eta, \delta)\neq 0$, the line and quadric intersect in one point.

(c) If $(Q\delta, \delta)=0$, $(A\eta, \delta)=0$, and $(A\eta, \eta)\neq 0$, the line does not intersect the quadric.

(d) If (3.2) is an identity, then the line lies on the quadric.

4. THEOREM (Diametral plane) *The midpoints of all chords of a quadric parallel to a given direction δ lie in a plane called the diametral plane corresponding to that direction.*

Proof. (Indeed this theorem has been proved in many ways; most of them quite tediously). Here we shall demonstrate the use of Section 3. Assume that $(Q\delta, \delta) \neq 0$, so that there will be two points of intersection [see Sec. 3 (a)], and let η end at the midpoint of the corresponding chord. Then the chord lies along the line $\xi = \eta + t\delta$. Now let t_1 and $t_2 = -t_1$ (since η ends at the midpoint of the chord) be the values of t corresponding to the ends of the chord. Since $t_1 + t_2 = 0$, we conclude from (3.2) that $(A\eta, \delta) = 0$. Thus η ends on the plane $(A\xi, \delta) = 0$.

5. COROLLARY. *If the diametral plane is orthogonal to corresponding direction δ in Section 4, then δ is a characteristic direction of Q.*

Proof. We easily see that $(A\delta, \eta) = 0$ can be written in the form $(Q\delta, \eta) + M = 0$, where M is a constant and $\eta = (x_1 \cdots x_n)$. The normal direction to $(A\delta, \eta) = 0$ is $Q\delta$ and δ lies in this direction. Hence $Q\delta = k\delta$, which proves the corollary.

In this case the plane $(A\delta, \eta) = 0$ is called a plane of symmetry for the quadric.

6. A center of a quadric. A vector $\eta = (x_{10}, \cdots, x_{n0})$ is said to end at a center of a quadric $(A\xi, \xi) = 0$ if for any ξ_1 satisfying $(A\xi_1, \xi_1) = 0$ there is a ξ_2 satisfying $(A\xi_2, \xi_2) = 0$ such that

$$\eta = \tfrac{1}{2}(\xi_1 + \xi_2). \tag{6.1}$$

Now to obtain a center let

$$\xi = \eta + t\delta \tag{6.2}$$

be a line through the end of η, where

$$\delta = (l_1, \cdots, l_n) = (x_1 - x_{10}, \cdots, x_n - x_{n0}).$$

The points of intersection of (6.2) and the quadric are obtained by

$$(Q\delta, \delta)t^2 + 2(A\eta, \delta)t + (A\eta, \eta) = 0, \qquad (Q\delta, \delta) \neq 0$$

[see (3.2)]. If (6.1) is true, then

$$(A\eta, \delta) = 0. \tag{6.3}$$

Now let

$$A = \begin{pmatrix} h_{11} & \cdots & h_{1n} & p_1 \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ h_{1n} & & h_{nn} & p_n \\ p_1 & \cdots & p_n & d \end{pmatrix}.$$

Then (6.3) can be written as

$$(x_{10} \cdots x_{n0}) \begin{pmatrix} h_{11} & \cdots & h_{1n} & p_1 \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ h_{1n} & & h_{nn} & p_n \\ p_1 & \cdots & p_n & d \end{pmatrix} \begin{pmatrix} l_1 \\ \cdot \\ \cdot \\ l_n \\ 0 \end{pmatrix} = 0.$$

This implies that

$$\alpha = \left(p_1 + \sum_{i=1}^{n} h_{1i}x_{i0}, \cdots, p_n + \sum_{i=1}^{n} h_{in}x_{i0} \right)$$

is orthogonal to $\delta = (x_1 - x_{10}, \cdots, x_n - x_{n0})$ for all $\xi = (x_1, \cdots, x_n)$ satisfying $(A\xi, \xi) = 0$. Either α is a fixed direction, and $(A\xi, \xi) = 0$ is a double plane perpendicular to α at the end point of η, or $\alpha = 0$, that is,

$$p_j + \sum_{i=1}^{n} h_{ji}x_{i0} = 0, \qquad j = 1, \cdots, n,$$

which means $(x_{10}, \cdots, x_{n0})$ is the solution of

$$(x_1 \cdots x_n 1) \begin{pmatrix} h_{11} & \cdots & h_{1n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ h_{1n} & & h_{nn} \\ p_1 & \cdots & p_n \end{pmatrix} = (0 \cdots 0).$$

Of course the problem of the existence of center is the problem of n equations and n unknowns. We only mention that if $\det Q \neq 0$, there is a unique center and in the case $\det Q = 0$ we have to consider other determinants for obtaining centers.

7. Vertex of a quadric without center. For a quadric to lack a center the rank k of the matrix Q must be less than n. Let $\alpha_1, \cdots, \alpha_k$ be k orthonormal characteristic directions of Q corresponding to k nonzero characteristic roots of Q, and the corresponding planes of symmetry be

$$(A\alpha_i, \xi) = 0, \qquad i = 1, \cdots, k. \tag{7.1}$$

By well-known results on reduction of quadratic forms, the quadratic terms of the original quadric are a linear combination of the squares of the first degree terms of the left members of the equations (7.1).

We consider the system of $k+1$ equations

$$(A\xi, \xi) = 0; \quad (A\alpha_i, \xi) = 0, \qquad i = 1, \cdots, k. \tag{7.2}$$

Since $(A\alpha_i, \xi)^2 = 0$, we may subtract some expression of the form $\sum_i c_i (A\alpha_i, \xi)^2$ from the equation of the quadric, leaving a linear equation, and the system

(7.2) reduces to

$$(A\alpha_i, \xi) = 0, \qquad i = 1, \cdots, k$$

$$(7.3) \qquad (A\xi, \xi) - \sum_k c_i(A\alpha_i, \xi)^2 = \sum_n b_i x_i + c = 0.$$

Since the k planes of symmetry in (7.1) are orthogonal, and $k<n$, the system (7.3) has rank either k or $k+1$. We consider three possible cases:

(a) If the rank is k and the system has solutions, the last equation is a consequence of the first k equations. Then the system $(A\alpha_i, \xi)=0$, $i=1, \cdots, k$, has solutions which are centers of the quadric, and lie in the quadric.

(b) If the rank is k, and there are no solutions, again the solutions of the system $(A\alpha_i, \xi)=0$, $i=1, \cdots, k$, are centers, but do not lie on the quadric.

(c) If the rank is $k+1$, the set of solutions is a hyperplane of dimension $n-k-1$. Any point of this hyperplane is called a vertex of the quadric. In particular, if $k=n-1$, there is a unique point which is the vertex of the quadric.

8. Polar of a point with respect to a quadric. Let the point be the end point of $\eta=(x_{10}, \cdots, x_{n0})$, and let δ be any direction. Then the equation of any line through the end point of η is

$$(8.1) \qquad \xi = \eta + t\delta.$$

The points of intersection of (8.1) and a quadric $(A\xi, \xi)=0$ are obtained by

$$(8.2) \qquad (Q\delta, \delta)t^2 + 2(A\eta, \delta)t + (A\eta, \eta) = 0$$

[see (3.2)]. Let $(Q\delta, \delta)\neq 0$. Then (8.2) has two solutions and (8.1) intersects $(A\xi, \xi)=0$ in two points. The locus of the harmonic conjugate of the end point of η with respect to these points is a plane, called the polar of the end point of η with respect to $(A\xi, \xi)=0$.

Proof. Let t_1 and t_2 be the roots of (8.2) and let t correspond to ξ which ends at the harmonic conjugate of the end point of η with respect to two points of intersection. Then

$$\frac{2}{t} = \frac{1}{t_1} + \frac{1}{t_2} \quad \text{or} \quad t = \frac{2t_1t_2}{t_1 + t_2} = -\frac{(A\eta, \eta)}{(A\eta, \delta)}.$$

If $(A\eta, \delta)=0$, then η ends at a center of $(A\xi, \xi)=0$ and the polar is a plane at infinity. Let $(A\eta, \delta)\neq 0$. Then we have

$$(A\eta, \delta)\xi = (A\eta, \delta)\eta - (A\eta, \eta)\delta.$$

Taking inner product of both sides by $A\eta$ we get $(A\eta, \delta)(A\eta, \xi)=0$. Since $(A\eta, \delta)\neq 0$ we get $(A\eta, \xi)=0$, which is a plane.

If η ends on $(A\xi, \xi)=0$, then this plane is the tangent plane at the end point of η.

COMPLEX GROUPS

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1. Introduction. The notion of quotient groups is usually arrived at through consideration of normal subgroups. In this paper, quotient groups are characterized by means of three postulates. The subsequent removal of two of these postulates gives rise to a class of mathematical systems, called complex groups, possessing greater set-theoretic generality than quotient groups. Since each complex group is itself a group, a technique for constructing new groups from a given group is implicitly involved.

2. Complexes. A *complex* of a group G is any subset of G. In particular, the null set $\varnothing$ is a complex of G. If A and B are complexes of G, then the *complex product* AB of A and B is the set of all elements ab, where $a \in A$ and $b \in B$. The formation of a complex product AB is termed *complex multiplication*, and A and B are called *factors* of AB. If a factor of a complex product is the null set, then the complex product itself is the null set. Braces are suppressed when a factor is a single-element set. Thus, for example, the complex product of $\{a\}$ and B is written aB instead of $\{a\}B$. The cardinal number of a complex B is denoted $c(B)$. A complex B is said to be *torsion* if each of its elements has finite period. B is *idempotent* if $BB = B$.

For the sake of completeness, a few of the elementary rules governing complex multiplication are presented. Notice that, in general, complex multiplication is not distributive over set-theoretic intersection. If A, B, and C are complexes of a group G, then

(i) $A(BC) = (AB)C$,
(ii) $A(B \cup C) = AB \cup AC$ and $(B \cup C)A = BA \cup CA$,
(iii) $A(B \cap C) \subseteq AB \cap AC$ and $(B \cap C)A \subseteq BA \cap CA$,
(iv) if $A \subseteq B$, then $CA \subseteq CB$ and $AC \subseteq BC$, and
(v) if $A \neq \varnothing$ and $B \neq \varnothing$, then $c(AB) \geqq c(A)$ and $c(AB) \geqq c(B)$.

3. Postulates for a quotient group. A set $\mathfrak{K}$ of complexes of a group G is a quotient group of G if and only if $\mathfrak{K}$ satisfies the following postulates:

(P1) $\mathfrak{K}$ is a group under complex multiplication,
(P2) $\mathfrak{K}$ covers G, and
(P3) members of $\mathfrak{K}$ are pairwise disjoint.

These postulates are consistent and independent. Of course, it must be verified that they actually characterize quotient groups.†

Let e and E denote respectively the identity elements of G and $\mathfrak{K}$. It will first be shown that E is a subgroup of G. Since $\mathfrak{K}$ covers G, there exists an $X \in \mathfrak{K}$

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† J. E. Eaton, making use of different definitions and concepts, has proved a theorem from which this would follow almost at once. Amer. J. Math., vol. 62, 1940, pp. 222–226.

such that $e \in X$. Clearly, $X \subseteq XX$, and in view of (P3), $X = XX$. But $X = XX$ implies that $X = E$, therefore $e \in E$. Now, suppose that $x \in E$ and $x^{-1} \in X \in \mathfrak{K}$. Since $xx^{-1} \in EX = X$, it follows from (P3) and the fact that $e \in E$ that $X = E$. Thus E contains the identity and the inverse of each of its elements, and is a subgroup of G.

To see that E is normal in G, notice that for each $x \in G$ there exists an $X \in \mathfrak{K}$ such that $x \in X$ and $xEx^{-1} \subseteq XEX^{-1} = E$. From the relations $xEx^{-1} \subseteq E$ and $x^{-1}Ex \subseteq E$ it easily follows that E is a normal subgroup of G.

It remains to show that each $X \in \mathfrak{K}$ is a left, and hence a right, coset of E. Let $x \in X \in \mathfrak{K}$, and observe that $x^{-1} \in X^{-1}$, $xE \subseteq XE = X$, and $x^{-1}E \subseteq X^{-1}E = X^{-1}$. The relation $xE = X$ ensues, and the demonstration is complete.

4. Complex groups. A *complex group from a group* G, or more simply, a *complex group*, is a set of complexes of G which satisfy (P1) but not necessarily (P2) or (P3). Thus, a complex group is merely a set of complexes which themselves form a group under complex multiplication. Obviously, any quotient group of any subgroup of G is a complex group from G.

The notation $\mathfrak{G}$ is used to denote any complex group from group G. In order to avoid confusion, elements of $\mathfrak{G}$ are called *members*. The symbols X, Y, and Z are used to denote members of $\mathfrak{G}$, while the symbol E is reserved for the identity member of $\mathfrak{G}$. Similarly, x, y, and z denote elements of G, and e denotes the identity element of G. The set-theoretic union of the members of $\mathfrak{G}$ is denoted by $\overline{\mathfrak{G}}$. A complex group is *disjunctive* if its members are pairwise disjoint. A complex group is *linearly ordered* if its members are pairwise comparable. A complex group is *finite* if it has finitely many members. Since complex groups are not necessarily abelian, the multiplicative notation for ordinary group theory is used. Thus, X^{-1} is the unique member of $\mathfrak{G}$ such that $XX^{-1} = X^{-1}X = E$, and as usual, $X^0 = E$ and $X^{-n} = (X^n)^{-1}$, where n is a positive integer.

Example 1. Let A be the additive group of integers, and let $\mathfrak{A}$ be the set of all complexes of A of the form $\{n\} \cup \{x \mid x \in A \text{ and } x \geqq n+5\}$, where n is an integer. $\mathfrak{A}$ is neither disjunctive nor linearly ordered, and the identity member $E = \{0\} \cup \{x \mid x \in A \text{ and } x \geqq 5\}$ is not a subgroup of A.

Example 2.* Let M be the multiplicative group of positive rational numbers, and let $\mathfrak{M}$ be the set of all complexes of M of the form $\{x \mid x \in M \text{ and } x > r\}$, where r is a positive real number. $\mathfrak{M}$ is linearly ordered, and the identity element 1 of M is not contained in the identity member $E = \{x \mid x \in M \text{ and } x > 1\}$ of $\mathfrak{M}$. Furthermore, since $\mathfrak{M}$ is nondenumerable, we see that it is possible for a group G to have a complex group $\mathfrak{G}$ which is isomorphic with no quotient group or subgroup of G.

Example 3. Let G be the group obtained by forming the direct product $A \otimes M$, where A and M are the groups defined in the previous two examples.

* I am indebted to the referee for this interesting example.

Construct in G the three complexes $K_1=\{(x, y)|x\in A$ and $y\in M$ and $y>1\}$, $K_2=\{(x, 1)|x\in A$ and x is odd$\}$, and $K_3=\{(x, 1)|x\in A$ and x is even$\}$. Set $E=K_1\cup K_3$ and $X=K_1\cup K_2$. The set $\mathfrak{G}$ whose members are E and X is a complex group from G. $\mathfrak{G}$ is finite and cyclic, but not disjunctive. Furthermore, $\mathfrak{G}$ is not isomorphic with any subgroup of G.

5. Preliminary results. It is clear from the examples that there exist complex groups which are not quotient groups. For convenience, a *komplex group* is defined to be a complex group which is not a quotient group. The first theorem and its corollaries indicate where one must look to find komplex groups.

THEOREM 1. *If $\mathfrak{G}$ is a complex group from group G, then $\mathfrak{G}$ is a quotient group of $\overline{\mathfrak{G}}$ if and only if E is a subgroup of G.*

Proof. It is sufficient to show that $\overline{\mathfrak{G}}$ and $\mathfrak{G}$ satisfy the conditions given for a quotient group in Section 3. First, it will be shown that $\mathfrak{G}$ is disjunctive. Suppose that X, $Y\in\mathfrak{G}$ and $X\neq Y$. Since $(XY^{-1})(YX^{-1})=E$, it follows that either $XY^{-1}\cup YX^{-1}\subseteq E$ or $(XY^{-1}\cup YX^{-1})\cap E=\varnothing$. The first alternative does not hold because $XY^{-1}\subseteq E$ if and only if $E\subseteq YX^{-1}$, and $XY^{-1}\neq E$. From the second alternative it follows that $X\cap Y=\varnothing$, since otherwise, $X\cap Y\neq\varnothing$ implies that $XY^{-1}\cap E\neq\varnothing$. $\mathfrak{G}$ is therefore disjunctive, and it remains only to show that $\overline{\mathfrak{G}}$ is a subgroup of G. Clearly, $\overline{\mathfrak{G}}$ is a semigroup with identity e. Let $x\in X\in\mathfrak{G}$, and let $y\in X^{-1}\in\mathfrak{G}$. Then $xy\in XX^{-1}=E$, and $x^{-1}=y(xy)^{-1}\in X^{-1}$. Thus, $\overline{\mathfrak{G}}$ contains the inverse of each of its elements, and the proof is complete.

A complex group $\mathfrak{G}$ from group G is *elementary* if each of its members is a finite complex of G. It is easy to show that all members of an arbitrary complex group have the same cardinal number, therefore a complex group is elementary if and only if each of its members contains n elements, where n is a fixed natural number.

COROLLARY 1. *Every elementary complex group is a quotient group.*

COROLLARY 2. *$\mathfrak{G}$ is a quotient group of $\overline{\mathfrak{G}}$ if E is torsion.*

COROLLARY 3. *$\mathfrak{G}$ is a torsion quotient group of $\overline{\mathfrak{G}}$ if $\overline{\mathfrak{G}}$ is torsion.*

Komplex groups can never be constructed from torsion groups. On the other hand, any group which is either mixed or torsion free possesses komplex groups. This is evident from Example 1 and the fact that any group which is not torsion contains an infinite cyclic subgroup.

Not every disjunctive complex group is a quotient group. This may be seen by returning to Example 2, and considering the complex group from M whose sole member is $E=\{x|x\in M$ and $x>1\}$. The next theorem gives a necessary and sufficient condition for a complex group to be disjunctive. A member of a complex group is said to be *isolated* if its intersection with each of the remaining members is null.

THEOREM 2. *A complex group $\mathfrak{G}$ from group G is disjunctive if and only if $\mathfrak{G}$ possesses an isolated member.*

Proof. Suppose that $X, Y, Z \in \mathfrak{G}$ such that $X \neq Y$ and Z is an isolated member. Since $Z \cap ZX^{-1}Y = \varnothing$, and $X \cap Y = \varnothing$ if and only if $(ZX^{-1})X \cap (ZX^{-1})Y = \varnothing$, it follows that $X \cap Y = \varnothing$. Therefore, $\mathfrak{G}$ is disjunctive, and the theorem is proved.

An interesting feature of complex groups which are not disjunctive is revealed by the next theorem.

THEOREM 3. *The intersection of two distinct members of a complex group is either null or infinite.*

Proof. Let $X, Y \in \mathfrak{G}$ such that $X \neq Y$ and $X \cap Y \neq \varnothing$. It follows that $E \cap X^{-1}Y \neq \varnothing$ and $Y^{-1}X \cap E \neq \varnothing$. Since $e \in X^{-1}Y \cap Y^{-1}X$, assume without loss of generality that $e \notin X^{-1}Y$, and hence that $e \in E \cap X^{-1}Y$. Now observe that for each positive integer n, $(E \cap X^{-1}Y)^n \subseteq E \cap X^{-1}Y$, because $(E \cap X^{-1}Y)^n \subseteq E \cap X^{-1}Y \cap (X^{-1}Y)^2 \cap \cdots \cap (X^{-1}Y)^n$. Thus each element in $E \cap X^{-1}Y$ has infinite period, and it follows that $X \cap Y$ is infinite, as was to be shown.

6. Finite complex groups. As might be expected from consideration of Example 3, the requirement that $\mathfrak{G}$ possess finitely many members does not guarantee simple set-theoretic structure for $\mathfrak{G}$. The best set-theoretic result for general finite complex groups is obtained as a corollary to the next theorem.

THEOREM 4. *Two distinct members of a torsion complex group $\mathfrak{G}$ are never comparable.*

Proof. Suppose that $X, Y \in \mathfrak{G}$ such that $X \subseteq Y$. Then $XY^{-1} \subseteq E$, and since $\mathfrak{G}$ is torsion, XY^{-1} has finite period n. But $(XY^{-1})^n \subseteq XY^{-1} \subseteq E$ implies that $XY^{-1} = E$ and $X = Y$. The proof is complete.

COROLLARY 4. *Two distinct members of a finite complex group are never comparable.*

Let $\mathfrak{G}$ be a complex group with identity member E. The *core* of $\mathfrak{G}$ is defined to be the set of all members of $\mathfrak{G}$ which have a nonnull intersection with E. It is easy to show that the core of $\mathfrak{G}$ is a subgroup of $\mathfrak{G}$. Occasionally, this fact can be used to help determine the structure of a complex group. For example, if $\mathfrak{G}$ is a finite complex group of prime order, then either $\mathfrak{G}$ is disjunctive or $\mathfrak{G}$ is equal to its core.

7. Maximal complex groups. Let $\mathfrak{G}$ be a complex group from group G with identity member E. If each complex group from G having E as its identity member is a subgroup of $\mathfrak{G}$, then $\mathfrak{G}$ is said to be *maximal*. The next theorem, which follows at once from the lemma, states that there is a one-to-one correspondence between the set of all idempotent complexes of G and the set of all maximal complex groups from G.

LEMMA 1. *If $\mathfrak{H}$ and $\mathfrak{K}$, are complex groups from group G having the same identity member E, then there exists a complex group $\mathfrak{G}$ from G such that $\mathfrak{H}$ and $\mathfrak{K}$ are subgroups of $\mathfrak{G}$, and E is the identity member of $\mathfrak{G}$.*

Proof. Define $\mathfrak{G}$ to be the set of all complex products of finitely many members of $\mathfrak{H} \cup \mathfrak{K}$. It is now an easy matter to verify that $\mathfrak{G}$ has the properties mentioned in the lemma.

THEOREM 5. *Associated with each idempotent complex K of a group G is a unique maximal complex group from G which has K for its identity member.*

8. Divisible properties. A property P is said to be *divisible* if whenever a group G possesses property P then all complex groups from G possess property P. The property of being abelian is an obvious divisible property. From Corollary 3 it may be seen that the property of being torsion is a divisible property. The final theorem mentions a third divisible property. The proof is omitted because, although straightforward, it is lengthy.

THEOREM 6. *The property of being cyclic is a divisible property.*

GULLIVER'S RETURN TO THE LAND WITHOUT ONE, TWO, THREE

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Gulliver's interest in advanced mathematics can be traced to adventures not mentioned by Swift and only recently recorded.* They took place on an island which, because of the current sophisticated symbols for numbers, the traveller called the Land without One, Two, Three. Upon his return to Europe, Gulliver studied, both in England and on the Continent, what then (that is, in the early 1700's) was the most modern topic of investigation: the functions

$$x, x^2, x^3, \sqrt{x}, \cos x, \log x, \cdots .$$

The more the traveller penetrated into analysis, the more he admired the ingenious theory. Yet he was puzzled by several features of those sophisticated symbols for functions, which somehow reminded him of the numerals in the Land without One, Two, Three—features of analysis that, incidentally, are still apparent today.

(a) Each symbol for a function† includes the letter x and is equivocal in that it also designates the value assumed by that function for x. For instance, the function $\log x$ assumes the value $\log x$ for x.

* Karl Menger, Gulliver in the land without one, two, three, Math. Gaz., vol. 43, 1959, pp. 241–250.

† The only exceptions are the constant functions. The same numeral designates the number 3 and the function assuming the value 3 for each number x.

(b) Power functions are denoted by symbols that differ in structure from those in which x is *preceded* by signs such as $\surd$, cos or log. In each symbol for a higher power, x is *followed* by a raised small numeral. The first power, also called *identity function*, is denoted simply by x.

(c) Symbols for the results of substitutions into a function, say into log x, include merely the x-free part of its symbol. For instance, mathematicians write log cos x and not log x(cos x) when substituting cos x into the function log x. Similarly, the x-free part of a function symbol is all that is used in applying the function to what Newton called *fluents*, *e.g.*, the time t and the gas pressure p; that is to say, mathematicians write, for instance, t^2 and log p, and not $x^2(t)$ or log $x(p)$.

(d) The symbol x, because of its equivocal use, is unfit to express in a simple way some of the most important properties of the identity function. For instance, the law that substitution of the function x into the function x yields the function x cannot be expressed by writing $xx=x$ or $x(x)=x$. In order to formulate it, one must ad hoc describe the identity function by a symbol that is better than x, say by $h(x)$ and write:

$$\text{If } h(x) = x, \quad \text{then} \quad h(h(x)) = h(x) \quad \text{or} \quad h(h(x)) = x.$$

This important law about the so-called function x cannot be rendered in a simple formula. What is needed is an implication. The same is true for certain left-side distributive laws involving substitution.§ They, too, can only be expressed in the form of implications, such as

$$\text{If } s(x) = f(x) + g(x) \quad \text{and} \quad p(x) = f(x)\cdot g(x),$$
$$\text{then } s(h(x)) = f(h(x)) + g(h(x)) \quad \text{and} \quad p(h(x)) = f(h(x))\cdot g(h(x)).$$

But rarely are those important laws mentioned at all. In comparison with addition and multiplication, analysis underemphasizes substitution, the functional operation par excellence, and ignores important laws about the functions possessing wayward symbols—the powers and the identity. These were the facts that struck Gulliver most.

Yet it appeared to him that the current theory of *functions of* x could easily be transformed into a (largely substitutive) theory of *functions*, namely, by

(1) introducing a permanent symbol, say j, for the identity function and $j^2, j^3, \cdots$ for the higher power functions—for the functions *themselves* and not for function *values*;

(2) shedding the x in symbols for other functions; in other words, by using function symbols such as j, j^2, j^3, $\surd$ or $j^{1/2}$, cos, and log.

(A) No function symbol would then include the letter x or be equivocal. Followed by x, it would unequivocally designate the value assumed by that function for x.

§ The corresponding right-side distributive laws are not valid for any three functions. For instance, there are functions such that $h(f(x)+g(x))\neq h(f(x))+h(g(x))$.

(B) The discrepancy between the symbols for power functions and other function symbols would disappear. The value that j^2 assumes for x might also be denoted by $j^2(x)$ or even j^2x, instead of by x^2, just as $\sqrt{x}$ and $\log x$ denote the values of other functions.

(C) The results of substitutions into a function f might be expressed by appending to the letter f the symbol for the function that is being substituted, for instance, by writing fg, $\log\cos$, $\cos j^2$, $j^2\cos$, j^2j^3. Similarly, instead of t^2, one might write j^2t for the square of the time in analogy to $\log p$ for the logarithm of the pressure.

Clearly, mere juxtaposition of function symbols must not be used to express both substitution and multiplication. If it denotes substitution, then in designating the product of functions one needs some sign, say a dot between the symbols for the factors, as in*

$$j^2\cdot j^3 = j^5 \quad \text{and} \quad \log\cdot\cos = \cos\cdot\log \text{ in contrast to } j^2j^3 = j^6 \quad \text{and} \quad \log\cos \neq \cos\log.$$

(D) A permanent symbol such as j would make it possible to express the basic properties of the identity function in simple formulas such as†

$$jj = j \quad \text{and} \quad jf = f = fj \qquad \text{for any function } f.$$

Transparent formulations of the left-side distributive laws involving substitution would read:

$$(f + g)h = fh + gh \quad \text{and} \quad (f\cdot g)h = fh\cdot gh \qquad \text{for any functions } f, g, h.$$

Then, at long last, those laws would appear on a well-deserved equal footing with the traditional additive-multiplicative distributive law§

$$(f + g)\cdot h = f\cdot h + g\cdot h \qquad \text{for any functions } f, g, h.$$

Gulliver was sensitive to the afore-mentioned difficulties concerning symbols for functions and ready to solve them because he had found the inhabitants of the Land without One, Two, Three experiencing difficulties with their symbols for numbers and he had proposed solutions by simplifying their system of numerals. After his study of analysis in Europe, the traveller, therefore, got curious about whatever had become of arithmetic on that island. So, the next time he was in the vicinity, several years after his first stay, Gulliver revisited the Land without One, Two, Three.

Most islanders still counted the way they had done before; that is to say,‡

* If juxtaposition denotes multiplication, then substitution must be indicated by some symbol, say, by $\circ$ or $\leftarrow$. For instance, one would write $j^3j^3 = j^6$ in contrast to $j^3 \circ j^3 = j^6$ or $j^3 \leftarrow j^3 = j^6$.

† If juxtaposition denotes multiplication, then the formulas about substitution would read $j \circ j = j$ or $j \leftarrow f = f = f \leftarrow j$ for any function f.

§ In conjunction with the commutative law of multiplication, this law implies its right-side counterpart $h\cdot(f+g) = h\cdot f + h\cdot g$ for any functions f, g, h.

‡ Cf. Math. Gaz., loc. cit. (footnote*, p. 641), p. 241.

$$|,\ |^{p},\ |^{t},\ 4|,\ 5|,\ 6|,\ \cdots$$

(pronounced: stix, stix-pair, stix-triple, four stix, five stix, six stix, · · ·). And, as before, the word *stix* (written |) also denoted the island's monetary unit.

Gulliver thus still was puzzled by the following features of the insular system:

(a′) Each symbol for a number† included the sign | and was equivocal in that it also designated a number of stix in the sense of monetary units. For instance, the symbol 5| indicated the number of such units in the amount of 5| as well as the amount itself.

(b′) The symbols for the three lowest numbers differed in structure from those in which | was *preceded* by signs such as 4, 5, 6. In the symbols for the second and third numbers, | was *followed* by small raised letters, while the first number was denoted simply by |.

(c′) The symbol for a product of two numbers included merely the stix-free part of the symbol of the first factor. For instance, arithmeticians wrote 4.5| (pronounced: four times five stix) and not 4|·5| (or four stix times five stix), which appeared inappropriate. Similarly, the stix-free part of a numeral was all that they used in counting objects other than stix; for instance, they wrote 5 fingers and eye^{p} (pronounced: five fingers and eye-pair) rather than 5| fingers and |^{p} eyes (or five stix fingers and stix-pair eyes), since this, too, would have looked (and sounded) awkward.

(d′) The symbol |, because of its equivocal use, was unfit to express in a simple way some of the most important properties of the number that it denoted. For instance, the law that the product of the number | and the number | is the number | could not be expressed by writing $|\cdot| = |$. *Stix times stix equals stix* would have sounded to the islanders as awkward as *dollar times dollar equals dollar* would to us. In order to formulate that law they had to introduce ad hoc a symbol that was better than |, say $h|$, and to write:

$$\text{If } h| = |, \quad \text{then} \quad h\cdot h| = h| \quad \text{or} \quad h\cdot h| = |.$$

This important law about the so-called number | could not be rendered in a simple formula. What was needed was an implication. The ordinary distributive law, too, could only be expressed in the form of an implication:

$$\text{If } s| = m| + n|, \quad \text{then} \quad s\cdot q| = m\cdot q| + n\cdot q|.$$

But rarely were those important laws mentioned at all—in line with the general neglect of multiplication, which, in contrast to addition, was taught only in advanced university courses.

During his first visit, Gulliver had come to the conclusion that what the island was lacking were symbols for the lowest numbers. (This was why he called it the Land without One, Two, Three.) In a way, even in the higher cases only

† The only exception was the number zero. The same numeral designated that number and the amount of no stix.

numbers of stix and not numbers themselves had symbols; and, indeed, many an insular arithmetical text was entitled *Theory of Numbers of Stix*. Yet he realized that the only requirement for the development of a *Theory of Numbers* would have been a minor notational change, namely,

(1′) introducing the symbols 1, 2, 3 for the three lowest numbers—for the numbers *themselves*, in contrast to numbers *of stix*;

(2′) shedding the stix in the higher numerals; in other words, using numerals

$$1, 2, 3, 4, 5, 6, \cdots.$$

(A′) No numeral would then include the sign | or be equivocal. Followed by |, it would designate the corresponding number of stix.

(B′) The discrepancy between the three lowest numerals and the others would disappear. The amounts of one stix, two stix, three stix (*i.e.*, monetary units), instead of by |, $|^p$, $|^t$, might also be denoted by 1|, 2|, 3|, just as 4| and 5| denoted larger amounts of monetary units.

(C′) The products of two numbers might be expressed by sandwiching a dot between the two numerals, as in $4\cdot5$, $5\cdot2$, $2\cdot5$, $2\cdot3$. Similarly, instead of eye^p, one might also write 2 eyes, in analogy to 5 fingers.

(D′) The symbol 1 would make it possible to express basic properties of the number one in simple formulas such as

$$1\cdot1 = 1 \quad \text{and} \quad 1\cdot n = n = n\cdot1 \quad \text{for} \quad \text{any number } n.$$

Similarly, a transparent formulation of the additive-multiplicative distributive law would read:

$$(m + n)\cdot q = m\cdot q + n\cdot q \quad \text{for} \quad \text{any numbers } m, n, q.$$

In such simple terms, multiplication would assume its proper place.

These ideas* Gulliver had propounded in a pamphlet *Arithmetic without Stix*, which circulated on the island widely, if, for various reasons, only in hand-written copies. The Immortals† were unimpressed and declared Gulliver's ideas to be trivial. The publishers' referees, on the contrary, complained about their abstruseness—some called stix-free numerals outright incomprehensible—and they all considered it as completely out of the question that, in terms of 1, 2, 3, numbers could ever be *taught*. Nor, obviously, could publishers regard a work whose title included the word *arithmetic* (in contrast, say, to the *Abstract Stix Theory and its Generalizations*) as a prestige book.

Toward the end of Gulliver's first stay, however, two little-known professors —mortal but alive—reported about experimental courses on stix-free multiplication. Their experiences had been satisfactory—in fact, so much so that they considered using the method in college courses. Approaching the pedagogical problem in a strictly scientific spirit, they designed large-scale experiments

* And other ideas; cf. Math., Gaz., loc. cit. (footnote*, p. 641).

† Immortals were the members of the all-powerful organization of the Incontestably Major Mathematicians of Real Talent and Learning.

which, they felt, should be arranged by the Insular Arithmetic Foundation—an organization sponsored by the island's ruler for various purposes including the improvement of arithmetical education. So they wrote to the director of the I. A. F. pointing out that Gulliver's ideas would make multiplication available to wider strata of the learning publication; but when Gulliver left they had not yet received an answer.

Even though on Gulliver's return to the Land without One, Two, Three most islanders still counted stix, stix-pair and so on, he found arithmetic in a state of fermentation. The island was swamped with beautifully printed new textbooks most of which discussed, in one form or another, the insular numerals and tried to improve them. (Even *Arithmetic without Stix* had been planographed.) Besides the symbol $n|$, there appeared various other notations for the amount of n monetary units, *e. g.*,

$$(n, |), \qquad (|, n), \qquad |n, \qquad \cdot|n.$$

"Whatever other difficulties may beset analysis," Gulliver thought, "the value that the function f assumes for x is denoted by $f(x)$; and, thank goodness, analysts have not to cope with other symbols such as

$$(f, x), \qquad (x, f), \qquad xf, \qquad \cdot xf."$$

(Of course, Gulliver was thinking of his contemporaries only, without anticipating all future developments.)

More important than those notational efforts was the fact that many new textbooks stressed the conceptual distinction between amounts of stix and numbers, *e.g.*, the *amounts* $|^p$ and $5|$, and the *numbers* $|^p$ and $5|$. Only some of the island's Major Mathematicians kept emphasizing that the traditional equivocation, being perfectly obvious, had never in the least confused any Immortal, whence they concluded in an impressive, if unusual, display of humility that it could not confuse anyone. Then there were a few who declared that using a symbol in discrepant meanings required courage which, they proclaimed, mathematicians ought to display.

Most mortal arithmeticians who had recognized the equivocal usage as such, however, tried to avoid it; and they attempted to express the distinction between numbers and numbers of stix in visible form. In fact, they tried very hard. Some reformers proposed to write $(|)^p$, $5(|)$, and $n(|)$ for the amounts and $(\cdot)^p$, $5(\cdot)$, and $n(\cdot)$ for the numbers. Others rendered the distinction by writing $|^p$, $5|$, $n|$ for the amounts, and $[|]^p$, $5[|]$, $n[|]$ for the numbers.

"Who back home would want to write $(x)^2$, $\log(x)$, and $f(x)$ for the values, and $(\cdot)^2$, $\log(\cdot)$, and $f(\cdot)$ for the functions?" Gulliver wrote in his diary. "Or x^2, $\log x$, $f(x)$ for the values, and $[x]^2$, $\log[x]$, $f[x]$ for the functions? If they wanted to improve things, they would rescue the identity function from its anonymity by introducing a symbol for it, say j. Thereafter they would write $j^2(x)$ or x^2, $\log x$, and $f(x)$ for the function values,

and j^2, log, and f for the functions. In fact, if they decided to discriminate between functions and numbers *typographically*, say, by using roman type for references to numbers, and italics or boldface letters for functions (as in j,^{2} *log*, f or **j**,^{2} **log**, **f**), then they might even save the parentheses in many symbols for function values and write j^2x, *log* x, and fx or $\mathbf{j}^2$x, **log** x, and **f**x."

Still other insular arithmeticians claimed that the words *five stix* (or 5|) were abbreviations of the words

the number of the amount of five stix (*or* 5|),

but that this phrase was so cumbersome that people got tired of repeating it and rather referred to the number, elliptically and with mental reservations, as *five stix*. Gulliver, on the other hand, was an outspoken man with an idiosyncrasy against mental reservations, particularly in mathematics. Besides, from his point of view, the number in question could be satisfactorily denoted by the single word *five*, wherefore the words *five stix* were redundant or hyperbolic rather than abbreviated and elliptic.

"Back home," the traveller wrote, "the symbol log x might be considered as an abbreviation or elliptical rendering of the phrase

the function whose value for x *is* log x.

But at the same time the symbol is also redundant and hyperbolic as a designation of the logarithmic function or the function log. Certain lacunae in a symbolism create *hyperbolic ellipses*."

A last group of reformers treated pure numbers as *ratios*. They replaced the traditional numerals 5| and $|^p$ by the symbols

5| $\rightarrow$ | and $|^p \rightarrow$ | (pronounced: five stix to stix and stix-pair to stix)—

a notation that they regarded as very suggestive, if in need of a supplementary explanation, namely, that

$$(|^p \rightarrow |)| = |^p \quad \text{and} \quad (|^p \rightarrow |) \text{ canoes} = \text{canoe}^p.$$

The followers of this line of thought trusted that the elegant symbols $|\rightarrow|$, $|^p\rightarrow|$, and $|^t\rightarrow|$ (for the numbers that Gulliver proposed to denote by the uncouth numerals 1, 2, 3) would undoubtedly be adopted as standard.

"Some of the proposals in the Land without One, Two, Three aiming at an improvement of the traditional numerals are sophisticated indeed," Gulliver wrote in his diary. "Their analogue in the realm of functions would be the proposal that, instead of designating the logarithmic and square and identity functions by simple symbols such as log and j^2 and j (and their values for x, by log x and $j^2(x)$ and $j(x)$), one should rather denote the functions by

$$x \to \log x \quad \text{and} \quad x \to x^2 \quad \text{and} \quad x \to x$$

and explain that

$$(x \to \log x)(x) = \log x \quad \text{and} \quad (x \to x^2)(\iota) = \iota^2 \quad \text{and} \quad (x \to x)(c) = c\text{—}$$

an inconceivable proposal."

(In this case, too, Gulliver spoke only about his contemporaries and failed to foresee all that would be written 250 years later.)

The two lively mortals still had not heard from the I. A. F., but had, of course, nonetheless gone ahead since they considered their educational experiments to be in the public interest. They wrote a second report which, in contrast to the first, was deemed worthy by the *Arithmetical Reviews* of a mention of its title. In this report the authors described how they made some bright youngsters in a high school (yes, high school!) learn something until then unheard of on the island—a multiplication table:

$$\begin{aligned} &1\cdot 1 = 1, \quad 1\cdot 2 = 2, \quad 1\cdot 3 = 3, \cdots \\ &2\cdot 1 = 2, \quad 2\cdot 2 = 4, \quad 2\cdot 3 = 6, \cdots \\ &3\cdot 1 = 3, \quad 3\cdot 2 = 6, \quad 3\cdot 3 = 9, \cdots \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}$$

Students trained in the traditional way, on the other hand, only in their senior year in college, caught a few glimpses of the advanced arithmetic of stix, such as:

$$\begin{aligned} &\text{If } h| = |, \quad \text{then } h\cdot h| = |, \quad h\cdot |^p = |^p, \quad h\cdot |^\iota = |^\iota, \cdots \\ &\text{If } h| = | \text{ and } k| = |^p, \quad \text{then } k\cdot h| = |^p, \quad k\cdot |^p = 4|, \quad k\cdot |^\iota = 6|, \cdots \\ &\text{If, moreover, } m| = |^\iota, \quad \text{then } m\cdot h| = |^\iota, \quad m\cdot |^p = 6|, \quad m\cdot |^\iota = 9|, \cdots \end{aligned}$$

Such profound results naturally could not be degraded to the level of memorizing material; nor were students expected to remember those sophisticated formulations.

Besides the new books that completely ignored the numerals 1, 2, 3, the two independent mortals showed Gulliver also some texts freely using them as though 1, 2, 3 had always been current. Besides, the discoverers of the celebrated formula $| + |^p = |^p + |$, who called *Arithmetic without Stix* trivial, had begun a reform of the insular symbolism—of course a nontrivial reform. The *Arithmetical Reviews*' report about their work was condensed in the *Insular Daily Press* under the headlines: OUR NUMERALS REVOLUTIONIZED. IMMORTALS INTRODUCE II AND III INSTEAD OF $|^p$ AND $|^\iota$.

The situation thus was not without promising signs. So, after a week's stay, the traveller wrote in his diary:

"Leaving the Land with embryonic One, Two, Three. Returning to the World without Identity and Power Functions."

ON NONUNIFORM DIFFERENTIABILITY

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In a recent paper [3], Robert Weinstock proved an interesting result regarding the continuity of the derived function $f'(x)$, where it is assumed that $f(x)$ possesses a finite differential coefficient at every point in its domain of definition. He proved that if the derivative $f'(x)$ exists on an open interval I of which $a \leqq x \leqq b$ is a closed subinterval, then a necessary and sufficient condition that $f'(x)$ be continuous on $a \leqq x \leqq b$ is that $f(x)$ be uniformly differentiable on $a \leqq x \leqq b$.

The main result which we propose to establish in this paper is embodied in the following

THEOREM. *If $f(x)$ is differentiable at every point of the open interval I, of which $[a, b]$ is a closed subset, then the set of points of nonuniform differentiability of $f(x)$ on $[a, b]$ is of first category and is an F_σ.*

To prove this result we introduce the notion of the points of "uniform" and "nonuniform" differentiability of $f(x)$ defined as follows:

Let

$$\phi(x, h) = \frac{f(x + h) - f(x)}{h} - f'(x).$$

By definition $f(x)$ is uniformly differentiable on $a \leqq x \leqq b$ if for each $\epsilon > 0$ there exists an x independent $\delta > 0$ such that $|\phi(x, h)| < \epsilon$ whenever $0 < |h| < \delta$ $(a \leqq x \leqq b)$ and $x + h \in I$. Since $f(x)$ is differentiable in $a \leqq x \leqq b$ we have that if $\epsilon > 0$ is preassigned, then to every point x of $[a, b]$ there corresponds a $\delta(x) > 0$ such that $|\phi(x, h)| < \epsilon$ whenever $|h| \leqq \delta(x)$. For a fixed $\epsilon > 0$ it may happen that $\delta(x)$ has no positive lower bound in $[a, b]$. Then we can find at least one point ξ of $[a, b]$ and a neighborhood of ξ in which $\delta(x)$ has no positive lower bound.

A point at which for each $\epsilon > 0$ there exists a neighborhood in which $\delta(x)$ has a positive lower bound is said to be a point of uniform differentiability of $f(x)$.

A point in a neighborhood of which $\delta(x)$ has no positive lower bound for some sufficiently small $\epsilon > 0$ is said to be a point of nonuniform differentiability of $f(x)$.

The above definitions are equivalent to the following:

A point ξ is a point of uniform or nonuniform differentiability of $f(x)$ according as $\phi(x, h)$ has or has not the unique double limit zero, as $x \to \xi$, $h \to 0$.

Let ξ be an arbitrary point in $[a, b]$ and $(\xi - d, \xi + d)$, $d > 0$, be a neighborhood of ξ; let p be any positive number. Let U be the least upper bound of $|\phi(x, h)|$ for $\xi - d \leqq x \leqq \xi + d$ and $|h| \leqq p$. We denote the greatest lower bound of

* The author is indebted to the referee for a number of very valuable suggestions.

U as d and p tend to zero independently by $\alpha(\xi)$. At a point of uniform differentiability $\alpha(x)=0$; if $\alpha(x)\neq 0$, then x is a point of nonuniform differentiability of $f(x)$. In the following we establish some properties of $\alpha(x)$ analogous to the corresponding properties of the function defining the measure of nonuniform convergence of a series [2].

THEOREM 1. *A necessary and sufficient condition that $f'(x)$ be continuous at ξ is that $\alpha(\xi)=0$.*

Assume first that $f'(x)$ is continuous at $x=\xi$ and suppose that $\alpha(\xi)=p>0$. Let $0<\epsilon<p$. Since $f'(x)$ is continuous at $x=\xi$, corresponding to this ϵ there is an $h_1>0$ such that

$$|f'(x_1)-f'(x_2)|<\epsilon \tag{1}$$

whenever x_1 and x_2 are points of $(\xi-h_1, \xi+h_1)$. Again, since $\alpha(\xi)=p$, in every neighborhood of ξ, for an arbitrary $h(\neq 0)$, there exists a point x and a number h', $0<|h'|\leqq|h|$, such that $\phi(x, h')>\epsilon$, *i.e.*,

$$\left|\frac{f(x+h')-f(x)}{h'}-f'(x)\right|>\epsilon,$$

$$|f'(x+\theta h')-f'(x)|>\epsilon, \qquad 0<\theta<1. \tag{2}$$

With the neighborhood $(\xi-h_1, \xi+h_1)$ already chosen with regard to (1), we may choose $|h|$ so small that x and $x+\theta h'$ are points of this neighborhood. But (2) is a contradiction of (1) and hence $\alpha(\xi)=0$.

Conversely, if $\alpha(\xi)=0$, then corresponding to any $\epsilon>0$, we can find a neighborhood of ξ such that $|\phi(x, h)|<\frac{1}{3}\epsilon$ for every point x of that neighborhood and for all h such that $|h|\neq 0$ is sufficiently small. Choose a particular value of h, say h_1, for which this is true. Then $|\phi(x, h_1)|<\frac{1}{3}\epsilon$ for every x of this neighborhood. Denote this neighborhood by D_1 and let $\psi(x)=\{f(x+h_1)-f(x)\}/h_1$. Then $\psi(x)$ is continuous at ξ. So, corresponding to the above $\epsilon>0$ it is possible to determine a neighborhood of ξ, say D_2, such that $|\psi(x_1)-\psi(x_2)|<\frac{1}{3}\epsilon$ for every x_1, x_2 of D_2.

Let $D=D_1\cap D_2$. Then D is a neighborhood of ξ. Let α_1, α_2 be any two points of D. Then $|f'(\alpha_1)-f'(\alpha_2)|<\epsilon$. Hence $f'(x)$ is continuous at ξ.

Theorem 1 may be restated as

THEOREM 2. *If $f'(x)$ exists in an open interval I of which $a\leqq x\leqq b$ is a closed subinterval then the points of discontinuity of $f'(x)$ in $a\leqq x\leqq b$ are precisely the points of nonuniform differentiability of $f(x)$ in $a\leqq x\leqq b$.*

It has been shown [1] that $\alpha(x)$ is upper semicontinuous in $[a, b]$. Now, since $\alpha(x)$ is upper semicontinuous, we have that for any positive number β the set of points of $[a, b]$ where $\alpha(x)\geqq\beta$ is closed.

THEOREM 3. *For arbitrary $\beta>0$, the closed subset of $[a, b]$ on which $\alpha(x)\geqq\beta$ is nondense in $[a, b]$.*

Proof. Let G_β be the set of points of $[a, b]$ at which $\alpha(x) \geqq \beta > 0$. Suppose, if possible, that G_β is not nondense in $[a, b]$. Then there exists at least one subinterval of $[a, b]$ at which G_β is everywhere dense and, since G_β is closed, every point of it is a point of G_β. Let us denote this interval by $[a', b']$ and let ξ be any point of this interval. Since $\alpha(\xi) \geqq \beta$, if $\beta_1 (< \beta)$ is any positive number and $\{\delta_n\}$ any positive null sequence, then there exists a point ξ' in some neighborhood of ξ contained in $[a', b']$ and a number h_1, where $|h_1| < \delta_1$, such that

$$\left| \frac{f(\xi' + h_1) - f(\xi')}{h_1} - f'(\xi') \right| > \beta_1.$$

Since $f'(\xi')$ exists, we can find an h_2, where $|h_2| < |h_1|$, such that

$$\left| \frac{f(\xi' + h_2) - f(\xi')}{h_2} - \frac{f(\xi' + h_1) - f(\xi')}{h_1} \right| > \beta_1.$$

Again, since $\{f(x+h_2) - f(x)\}/h_2$ and $\{f(x+h_1) - f(x)\}/h_1$ are continuous at ξ', there exists a neighborhood D_1 of ξ' contained in $[a', b']$ such that for every point x of D_1 we have

$$\left| \frac{f(x + h_2) - f(x)}{h_2} - \frac{f(x + h_1) - f(x)}{h_1} \right| > \beta_1.$$

Now, D_1 is a neighborhood of ξ' and $\alpha(\xi') \geqq \beta$. So, corresponding to the positive number δ_2', where $\delta_2' = \min [\delta_2, |h_2|]$, there exists a point ξ'' in D_1 and a number h_3, where $|h_3| < \delta_2'$, such that

$$\left| \frac{f(\xi'' + h_3) - f(\xi'')}{h_3} - f'(\xi'') \right| > \beta_1.$$

In a similar manner, we can find a neighborhood D_2 of ξ'' contained in D_1 such that for every point x of D_2

$$\left| \frac{f(x + h_3) - f(x)}{h_3} - \frac{f(x + h_4) - f(x)}{h_4} \right| > \beta_1,$$

where $|h_4| < |h_3|$.

If the process is continued we get a decreasing sequence of neighborhoods D_m such that for every point x of D_m

$$\left| \frac{f(x + h_{2m-1}) - f(x)}{h_{2m-1}} - \frac{f(x + h_{2m}) - f(x)}{h_{2m}} \right| > \beta_1,$$

where $|h_{2m}| < |h_{2m-1}| < \delta_m'$ and $\delta_m' = \min [\delta_m, |h_{2m-2}|]$.

The neighborhoods D_m can be so chosen that there exists a point η in all of them and at this point

$$\left|\frac{f(\eta + h_{2m-1}) - f(\eta)}{h_{2m-1}} - \frac{f(\eta + h_{2m}) - f(\eta)}{h_{2m}}\right| > \beta_1$$

holds for every m.

Now, $\{h_{2m}\}$ is a null sequence and so from the above we arrive at a contradiction to the fact that $f(x)$ is differentiable at η. Hence G_β must be nondense in $[a, b]$.

Let $\{\beta_n\}$ be any positive null sequence. Then any point of nonuniform differentiability of $f(x)$ will belong to the set $\alpha(x) \geqq \beta_n$ for some n and this set is nondense in $[a, b]$. Hence the points of nonuniform differentiability of $f(x)$ are points of a set which is the outer limiting set of a sequence of nondense closed sets. Hence we obtain

THEOREM 4. *If $f(x)$ is differentiable at every point of the open interval I, of which $[a, b]$ is a closed subset, then the set of points of nonuniform differentiability of $f(x)$ on $[a, b]$ is of first category and is an F_σ.*

In conclusion, I express my earnest gratitude to Dr. H. M. Sengupta, for his kind help in the preparation of the paper.

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ON PERIODICITIES OF CERTAIN SEQUENCES OF RESIDUES

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Let H be the class of polynomials $F(x)$ such that $F(m)$ is an integer whenever m is an integer. It is not difficult to show that H is identical with the class of polynomials of the form

$$F(x) = \sum_{j=0}^{K} a_j \binom{x}{j},$$

where a_j is an integer for $0 \leqq j \leqq K$. Let M be an arbitrary positive integer:

$$M = \prod_{i=1}^{n} P_i^{k_i},$$

where $P_1, \cdots, P_n$ are distinct primes and $k_i \geqq 1$ $(1 \leqq i \leqq n)$. Let K be a positive

integer, and $r_1, \cdots, r_n$ be integers chosen so that

$$P_i^{r_i-1} \leqq K < P_i^{r_i} \qquad (1 \leqq i \leqq n). \tag{1}$$

Finally, define

$$N = \prod_{i=1}^{n} P_i^{k_i+r_i-1}.$$

The symbols M, N, and K will retain these meanings throughout the paper.

Since $\Delta^{K+1}F(m)=0$ for every integer m,* it is clear that the sequence $\{F(m) \pmod M\}$ must have a period which does not exceed M^K. It is our purpose to determine the exact periods of such sequences.

LEMMA 1. *The sequence* $\left\{\binom{m}{K} \pmod M\right\}$ $(m = \cdots, -1, 0, 1, \cdots)$ *is periodic with fundamental period* N.

Proof. Expand $\binom{m+N}{K}$ about $N=0$ to obtain

$$\binom{m+N}{K} = \binom{m}{K} + \sum_{j=1}^{K} \binom{N}{j}\binom{m}{K-j}. \tag{2}$$

We first show that $\binom{N}{j}$ is divisible by M for $j=1, \cdots, K$. Write

$$\binom{N}{j} = \frac{N}{j} \cdot \frac{N-1}{1} \cdot \frac{N-2}{2} \cdots \frac{N-(j-1)}{j-1}. \tag{3}$$

If $1 \leqq R \leqq j-1$, the highest power of P_i which divides R is $P_i^{r_i-1}$, since $j \leqq K < P_i^{r_i}$. Hence, since $P_i^{r_i-1}$ is a factor of N, $N-R$ is divisible by $P_i^{q_i}$ $(q_i \leqq r_i-1)$ if and only if R is. On the other hand, since j is not divisible by $P_i^{r_i}$, the factor $P_i^{k_i}$ in the numerator of N/j is not cancelled. As this argument holds for $i=1, \cdots, n$, we can conclude that

$$\binom{N}{j} \equiv 0 \pmod M \qquad (j = 1, \cdots, K),$$

and the periodicity follows from (2).

If N' is a second period, then the greatest common divisor of N and N' is also a period. Let $(N, N') = \prod_{i=1}^{n} P_i^{q_i}$. If N is not the fundamental period, then there is a subscript i such that $q_i < r_i + k_i - 1$. Without loss of generality, assume that $i=1$. Then

$$N_1 = P_1^{r_1+k_1-2} \prod_{i=2}^{n} P_i^{r_i+k_i-1}$$

is also a period. We will show that this is impossible for any $K \geqq P_1^{r_1-1}$. First we assert that

* Δ is the forward difference operator. That is, $\Delta F(m) = F(m+1) - F(m)$.

$$\binom{N_1}{P_1^{r_1-1}} \not\equiv 0 \pmod{M}. \tag{4}$$

This is obvious if $r_1=1$. If $r_1>1$, expand as in (3). For $R=1, 2, \cdots, P_1^{r_1-1}$, it can again be seen that the powers of P_1 in factors of the form $(N_1-R)/R$ are cancelled. Now $N_1/P_1^{r_1-1}$ is not divisible by $P_1^{k_1}$, and (4) follows.

Next let $K=P_1^{r_1-1}+j$ with $j>0$. If $\left\{\binom{m}{K} \pmod{M}\right\}$ has period N_1, it follows that

$$\binom{N_1+\nu}{P_1^{r_1-1}+j} \equiv 0 \pmod{M} \qquad (\nu = 0, 1, \cdots, j).$$

However, this leads to a contraction of (4), because we could then write

$$\binom{N_1}{P_1^{r_1-1}} = \sum_{\nu=0}^{j} (-1)^{j-\nu}\binom{N_1+\nu}{P_1^{r_1-1}+j}\binom{j}{\nu} \equiv 0 \pmod{M}.$$

This completes the proof of Lemma 1.

We can immediately generalize to

THEOREM 1. *Let*

$$F(x) = \sum_{j=0}^{K} a_j \binom{x}{j} \tag{5}$$

be in H. If $(a_K, M)=1$, *the sequence* $\{F(m) \pmod{M}\}$ *is periodic, with fundamental period N.*

Proof. If K is the least integer which satisfies (1), we can infer from Lemma 1 that $\{a_K\binom{m}{K} \pmod{M}\}$ has fundamental period N, while all lower degree terms have periods which are proper divisors of N. Thus the conclusion follows for this case. Assume that $K-r-1$, $(r\geqq 0)$, is the least integer which satisfies (1), and that the theorem is true when $K-r$ is the smallest such integer. Consider

$$\Delta F(x) = F(x+1) - F(x) = \sum_{j=0}^{K-1} a_{j+1}\binom{x}{j}.$$

In this equation, $K-1$ plays the role of K in (5). From the induction assumption, the sequence $\{\Delta F(m) \pmod{M}\}$ has the fundamental period N. From this it follows that the fundamental period of $\{F(m) \pmod{M}\}$ is not less than N, while from Lemma 1 it follows that it is not greater than N.

COROLLARY. *If* $F(x)$ *is any polynomial in H, of degree K, then the sequence* $\{F(m) \pmod{M}\}$ *has a fundamental period of the form*

$$N_1 = \prod_{i=1}^{n} P_i^{j_i}, \quad \textit{where} \quad 0 \leqq j_i \leqq r_i + k_i - 1 \ (i = 1, \cdots, n).$$

As a partial converse to Theorem 1, we have

THEOREM 2. *Let* $\{f_m\}$ $(-\infty < m < \infty)$ *be a sequence of integers, and let*

$$\Delta^{K+1} f_m \equiv 0 \pmod{M} \qquad (m = 0, \pm 1, \pm 2, \cdots).$$

Then there is in H *a polynomial* $F(x)$ *of degree not exceeding* K, *such that*

$$F(m) \equiv f_m \pmod{M} \qquad (m = 0, \pm 1, \pm 2, \cdots). \tag{6}$$

Consequently, the sequence $\{f_m \pmod{M}\}$ *has a fundamental period which divides* N.

Proof. Define

$$F(x) = \sum_{r=0}^{K} \Delta^r f_0 \binom{x}{r}.$$

Then

$$F(m) = f_m \qquad (m = 0, 1, \cdots, K). \tag{7}$$

Since $F(x)$ is of degree not greater than K, we have

$$0 = \Delta^{K+1} F(m) \equiv \Delta^{K+1} f_m \pmod{M} \qquad (-\infty < m < \infty),$$

and (6) follows from (7) by a trivial induction.

It can also be stated that, if in addition to the hypothesis of Theorem 2, there is an integer m such that $(\Delta^K f_m, M) = 1$, then the fundamental period of $\{f_m \pmod{M}\}$ is precisely N.

In the case where M is a prime, we can obtain a stronger result.

THEOREM 3. *Let* P *be a prime, and* $F(x)$ *a polynomial in* H, *of degree* K, *such that the coefficient of* $\binom{x}{K}$ *is not divisible by* P. *Then, if* $P^{r-1} \leqq K < P^r$, *the sequence* $\{F(m) \pmod{P}\}$ *is periodic with fundamental period* P^r. *Conversely, if a sequence of integers* $\{f_m\}$, $(-\infty < m < \infty)$, *is such that* $\{f_m \pmod{P}\}$ *has fundamental period* P^r, *then there is a polynomial* $F(x)$ *in* H, *with* $P^{r-1} \leqq \deg F(x) < P^r$, *and*

$$F(m) \equiv f_m \pmod{P} \qquad (-\infty < m < \infty). \tag{8}$$

Proof. The first statement is a special case of Theorem 1. For the converse, let $\{f_m \pmod{P}\}$ have the assumed periodicity, and consider the linear system in the P^r unknowns $\{a_i\}$:

$$\sum_{n=0}^{m} a_n \binom{m}{n} \equiv f_m \pmod{P} \qquad (m = 0, 1, \cdots, P^r - 1).$$

Since this is a diagonal system, with coefficients on the diagonal equal to unity, there is a unique solution $\{a_n\}$ in the field of integers modulo P. Define

$$F(x) = \sum_{n=0}^{P^r-1} a_n \binom{x}{n}.$$

Then $F(x)$ satisfies (8) for $m=0, 1, \cdots, P^r-1$. By the corollary to Theorem 1, $\{F(m) \pmod P\}$ also has period P^r, and therefore $F(x)$ satisfies (8) for all m. Deg $F(x) \geqq P^{r-1}$, since if not $\{F(m) \pmod P\}$ would have period P^{r-1}, and so would $\{f_m \pmod P\}$, contrary to assumption.

For an alternate proof of the converse, one can observe that $\Delta^{P^r} f_m \equiv 0 \pmod P$ for all m, and the conclusion essentially follows from Theorem 2.

Acknowledgement. The work reported here was stimulated by an attempt to prove [1], which follows from Theorem 3 for $P=2$.

The referee has pointed out that Lemma 1 has appeared previously in [2]. The result is obtained there for $m=0, 1, \cdots$, by a lengthy argument involving a chain of six lemmas and two theorems which are weaker than Lemma 1.

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THE DEFINITE INTEGRAL SYMBOL

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Smith [1] states that Leibniz refers to "the integral calculus as the *calculus summatorius*, a name connected with the summa (s) sign." Child [2] has directed attention to the invention of the $\int$ by Leibniz. Histories of mathematics [1, 3, 4], encyclopedias [5, 6] and some textbooks refer to the symbol $\int$ as an extended or elongated S, as a medieval S, as an old-fashioned S, the first letter of the word summa or sum, meaning the total amount, summation, the whole, etc. [7, 8, 9, 10]. Others ignore explanation of its morphology, etymology, or derivation [11, 12, 13, 14, 15].

Since Leibniz is generally credited with inventing the symbolism of modern calculus [1, 2, 6], I cannot refrain from remarking about an alternative derivation of the integral sign; one equally plausible as that generally stated.

Almost every botanist knows that if an apple is peeled by hand with a knife, beginning at the stem end and circling about the central axis without breaking the peel until the opposite end is reached, a regular spiral is obtained which forms an elongated S when placed on a flat surface with the inside of the peel up. Integration of the surfaces of certain solids having the shape of any of the three classes of spheroids, spherical, oblate, or prolate, yield such elongated S's, and continuous peelings of many fruits and melons that can be classified as spheroids take on the shape of the integral sign. Of these classes, all three may be found naturally occurring among citrus fruit, and the representative of any class which when peeled as shown in Figure 1 presents a satisfactory medieval S.

Now Bell [3] says of Leibniz (1646–1716), that "much of his mathematics,

to say nothing of his other wonderings . . . was written out in the jolting, draughty rattletraps that bumped him over the cow trails of the 17th century Europe . . . at his employers' erratic bidding." And, our own experiences in travel suggest that such conditions favor a firm acquaintanceship with apples. For what could have afforded a more convenient lunch to obtain, to carry or to prepare under such circumstances? Or, a more interesting one to contemplate when the road was smooth enough to permit peeling the apple? And the apple may have been bought beforehand in a convenient market for "by the end of the 17th century, fruit for market was recognized as a definite industry" [16] or picked fresh from cultivated apple trees, or wild apple trees indigenous to central Europe [17].

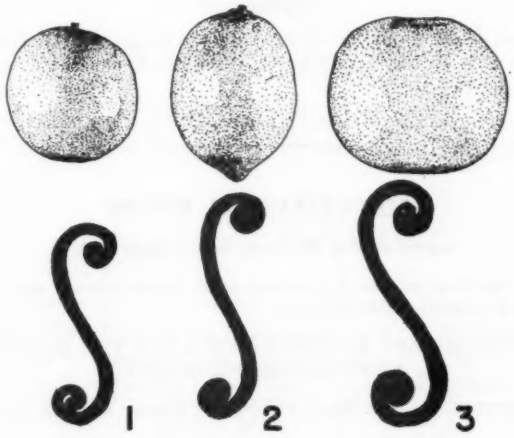

FIG. 1. Tracing of photographs of three kinds of citrus fruit and their respective peelings. 1, spherical Valencia orange; 2, prolate Eureka lemon; 3, oblate Marsh grapefruit.

The late Bing C. Wong, Professor of Mathematics at the University of California, Berkeley, I am informed by one of his former students, may have envisioned a similar derivation of the integral sign. But, I could not verify this through those of his associates I could reach.

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CORRECTION

W. R. Andress, *Basic properties of pandiagonal magic squares*, this MONTHLY, vol. 67, 1960, pp. 143–152. In the second line of Theorem 2, page 146, $a_{r+2,s+2}=a_{r,s}$ should be replaced by $a_{r,s}+a_{r+2,s+2}=0$.

MATHEMATICAL NOTES

EDITED BY ROY DUBISCH, Fresno State College

Because of the large number of papers on hand, consideration of new papers for this department has been temporarily suspended.

A DETERMINANT REPRESENTATION FOR THE CLASSICAL ORTHOGONAL POLYNOMIALS

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It is well known [1] that for each of the classical families of orthogonal polynomials (Hermite, Laguerre and Jacobi) there is a generalized Rodrigues formula through which the Nth member of the family is given (except for a normalization factor) by the relation

$$P_N = \frac{1}{w}\frac{d^N}{dx^N}(wF^N).$$

A particular family of polynomials is characterized by the choice of the functions w and F. For the Hermite polynomials, $w=e^{-x^2}$ and $F=1$; for the (generalized) Laguerre polynomials $w=e^{-x}x^\alpha$; and, for the Jacobi polynomials

$$w = (1-X)^\alpha(1+X)^\beta \quad \text{and} \quad F = 1-X^2.$$

It was shown [2] that the Nth derivative of a function $f(x)$ is given by

$$\frac{d^N f(x)}{dx^N} = \Delta_N f(x),$$

where the symbol Δ_N denotes the Nth ordered determinant

$$\begin{bmatrix} D_1 & -1 & 0 & 0 & 0 \cdots 0 \\ D_2 & D_1 & -2 & 0 & 0 \cdots 0 \\ D_3 & D_2 & D_1 & -3 & 0 \cdots 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & 1-N \\ D_N & \cdots & \cdots & D_3 & D_2 \quad D_1 \end{bmatrix}$$

in which the elements $D_1, \cdots, D_N$ are defined by

$$D_K = \frac{1}{(K-1)!} \frac{d^K \log f(x)}{dx^K}.$$

Upon using this formula for evaluation of the Nth derivative which occurs in the Rodrigues formula, one obtains $P_N = F^N \Delta_N$ where the elements of Δ_N are given by

$$D_K = \frac{1}{(K-1)!} \frac{d^K}{dx^K} [\log w + N \log F].$$

Now, it is easily shown by induction that when a determinant of the form Δ_N is multiplied by F^N, the result is the same as that of multiplying each element D_K by F^K. Thus it is seen that the Nth member of a family of orthogonal polynomials is given by $P_N = \Delta_N$ where the elements of Δ_N are defined by

$$D_K = \frac{F^K}{(K-1)!} \frac{d^K}{dX^K} [\log w + N \log F].$$

For the Hermite polynomials, one obtains

$$D_1 = -2x, \quad D_2 = -2, \quad \text{and} \quad D_K = 0 \quad \text{for} \quad K > 2;$$

for the (generalized) Laguerre polynomials,

$$D_1 = N + \alpha - X \quad \text{and} \quad D_K = (-1)^{K-1}(N + \alpha) \quad \text{for} \quad K > 1;$$

for the Jacobi polynomials, $D_K = -[(N+\alpha)(x+1)^K + (N+\beta)(x-1)^K]$. Thus it is seen that the classical orthogonal polynomials may be expressed in an especially simple way in terms of the determinant Δ_N.

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A NOTE ON EVALUATING DETERMINANTS*

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In a recent note, Lotkin [1] compares two methods for the evaluation of determinants, namely "the pivotal method" and "the method of contractants"; see [1] for details. He asserts that the method of contractants is somewhat of an improvement over the pivotal method. We compare in this note the method of contractants with the well-known very natural triangular method; see Scarborough [2], pp. 498–501. In the following table we list the formulas for the number of arithmetic operations, as stated by Lotkin, for the method of contractants and the corresponding formulas, easily verified, for a straightforward application of the triangular method.

Operation	*Contractants*	*Triangular*
Addition	$n(n-1)(2n-1)/4$	$n(n-1)(2n-1)/6$
Multiplication	$n(n-1)(2n-1)/3$	$n(n+1)(2n+1)/6-1$
Division	$(n-2)(n-1)(2n-3)/6$	$(n-1)(n+2)/2$

Let $C+$, $C\times$, $C\div$ be the number of respective operations for contractants acting on a determinant of order n, and let $T+$, $T\times$, and $T\div$ be the numbers for the triangular method. We note the following observations:

1. *$C+$ is asserted in [1] to be approximate.*
2. *$T+$ is exactly $\frac{2}{3}$ of $C+$.*
3. *$T\times$ is asymptotically $\frac{1}{2}$ of $C\times$.*
4. *$T\div$ is quadratic in n while $C\div$ is cubic.*
5. *The numbers of C operations are considerably increased by the presence of zeros in the given determinant, while the numbers of T operations are considerably decreased.*
6. *On new fast computers such as STRETCH and LARC, division takes about three times as long as multiplication.*

We conclude by listing for $n=10$, 50, and 100 the numbers of operations for these two methods. We do not assert that there do not exist methods faster than the triangular method.

n	$C+$	$T+$	$C\times$	$T\times$	$C\div$	$T\div$
10	427½	285	570	384	204	54
50	60,637½	40,425	80,850	42,924	38,024	1274
100	492,525	328,350	656,700	338,349	318,549	5049

References

1. M. Lotkin, Note on the method of contractants, this MONTHLY, vol. 66, 1959, pp. 476–479.

2. J. B. Scarborough, Numerical Mathematical Analysis, 3rd ed., Baltimore, 1955.

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SUCCESSIVE APPROXIMATION APPLIED TO QUADRATURE FORMULAS*

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The conventional treatment of quadrature formulas emphasizes a discrete nature of the independent variable. In most cases, however, it is desirable to interpret the increment as a continuous variable also. Indeed, error estimates generally are established using the latter interpretation. Quite naturally, one is then led to differential equation concepts rather than to mere quadrature concepts and the former may suggest alternative techniques. For example, the familiar Simpson's rule may be used to define an operator S, where

$$Sf(x) = \tfrac{1}{6}x[f(0) + 4f(\tfrac{1}{2}x) + f(x)], \tag{1}$$

which in turn yields a continuous approximation to the solution of the differential equation

$$\frac{dy}{dx} = Dy = f(x) \tag{2}$$

subject to the boundary condition, $y=0$ for $x=0$.

If we denote by y_1, the function defined in (1) and by y_2, the difference between the exact solution of (2) and the approximate y_1, then y_2 satisfies the differential equation

$$\frac{dy_2}{dx} = (1 - DS)f(x). \tag{3}$$

Since $y=y_1$ for $x=0$, (3) is subject to the boundary condition, $y_2=0$ for $x=0$. Thus (3) depicts the *growth* of the error in the approximate y_1 from its initial value 0 for $x=0$.

However, (3) is also a differential equation not unlike (2) itself and one might apply the operator S to obtain an estimate of the error y_2. Such a procedure is reminiscent of the familiar Picard iterative process of successive approximation used in the study of differential equations. Symbolically, the estimate of the error is given by $y_3=S(1-DS)f(x)$ and the sum y_1+y_3 might be expected to be an improvement over y_1. If we put $y_4=y_2-y_3$, then $y=y_1+y_3+y_4$, where now y_4 is the error and is seen to satisfy the differential equation $dy_4/dx=(1-DS)^2f(x)$. More generally, if for each positive integer j we denote by y_{2j} the solution of the differential equation

$$\frac{dy_{2j}}{dx} = (1 - DS)^j f(x), \tag{4}$$

subject to the boundary condition $y_{2j}=0$ for $x=0$, and if we denote by y_{2j+1}, the Simpson's rule approximation to the solution of (4), *i.e.*,

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$$y_{2j+1} = S(1 - DS)^j f(x), \tag{5}$$

then the exact solution of (2) may be expressed as

$$y = \sum_{j=0}^{k-1} y_{2j+1} + y_{2k}, \qquad k = 1, 2, \cdots, \tag{6}$$

where y_{2k} is interpreted as the error after $k-1$ iterations of the Simpson's rule operator. Note that (6) may be expressed as

$$y = S \sum_{j=0}^{k-1} (1 - DS)^j f(x) + y_{2k} \tag{7}$$

and if, as might be anticipated, $\lim_{k\to\infty} y_{2k} = 0$, (7) suggests an explicit solution of the form

$$y = S \sum_{j=0}^{\infty} (1 - DS)^j f(x). \tag{8}$$

If we formally "sum" the geometric series

$$\sum_{j=0}^{\infty} (1 - DS)^j = \frac{1}{1 - (1 - DS)} = \frac{1}{DS},$$

then (8) becomes $y = S(DS)^{-1} f(x)$ which, by a proper interpretation of the operators and the order in which they apply, yields the exact solution $y = D^{-1} f(x) = If(x)$. Clearly, what is required to justify (8), however, is the condition $\lim_{k\to\infty} y_{2k} = 0$.

An interesting illustration of the above is afforded by the power function $f(x) = x^n$, $n \geqq 0$. It is easily verified that for $n > 0$,*

$$(1 - DS)^j x^n = [1 - \tfrac{1}{6}(n+1)(1 + 2^{2-n})]^j x^n, \qquad j = 0, 1, \cdots$$

and thence by (4) and (5) (the latter together with (1)),

$$y_{2j} = [1/(n+1)][1 - \tfrac{1}{6}(n+1)(1 + 2^{2-n})]^j x^{n+1},$$
$$y_{2j+1} = [(1 + 2^{2-n})/6][1 - \tfrac{1}{6}(n+1)(1 + 2^{2-n})]^j x^{n-1}.$$

Thus it follows that

$$y = \tfrac{1}{6}(1 + 2^{2-n})(1 + e_n + \cdots + e_n^j) x^{n+1} + [1/(n+1)] e_n^{j+1} x^{n+1},$$

where $e_n = 1 - \frac{1}{6}(n+1)(1 + 2^{2-n})$ and clearly $\lim_{j\to\infty} y_{2j} = 0$† if and only if

$$|e_n| < 1. \tag{9}$$

When (9) is satisfied, the exact solution of (2) is indeed given by the geometric series

* For $n = 0$ we define $x^n \equiv 1$ and thus $(1 - DS)^j x^0 \equiv 0$.

† Uniformly in some finite interval, say, $0 \leqq x \leqq X$.

$y = \frac{1}{6}(1 + 2^{2-n})(1 + e_n + \cdots)x^{n+1} = \frac{1}{6}(1 + 2^{2-n})(1 - e_n)^{-1}x^{n+1} = [1/(n+1)]x^{n+1}$

and the above formalism is justified. On the other hand, (9) requires that n be less than 10.96 (approx.).

It is important to note that the fact of convergence does not depend on the range (or increment) of integration. In fact, e_n is the *relative* error inherent in this application of Simpson's rule (*i.e.*, the ratio of error to exact integral) and is a constant independent of the increment. (This illustrates the fallacy in a notion that quadrature formulas yield arbitrarily good results if the increment of the independent variable is chosen sufficiently small.) For $n > 10.96$, the absolute value of the relative error exceeds unity and the iterates, using Simpson's rule, merely introduce larger and larger errors.

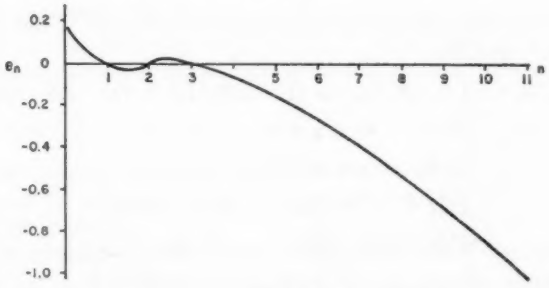

FIG. 1

Figure 1 is a plot of the relative error versus the exponent for the select range. At $n = 0$, the relative error is not continuous but jumps from zero to the limiting value $\frac{1}{6}$. This plot illustrates the well-known fact that Simpson's rule integrates polynomials of degree 3 or less exactly, *i.e.*, $e_n = 0$ for $n = 0$, 1, 2, and 3. Probably not so well known are the evidently good results for fractional exponents in the interval, $1 \leqq n \leqq 3$.

Finally, let us examine the case where $f(x)$ is a simple step function (*i.e.*, a step function with at most a finite number of steps). Since the Simpson's rule operator is additive, it will be sufficient to consider a function with but one step, say, $f(x) = 0$ for $0 \leqq x < a$ and $f(x) = c$ for $x > a$. The exact, continuous integral of $f(x)$ is given by $y = 0$ for $0 \leqq x < a$ and $y = c(x - a)$ for $x > a$, while the Simpson's rule approximation becomes $y_1 = 0$ for $0 \leqq x < a$, $y_1 = \frac{1}{6}cx$ for $a < x < 2a$, and $y_1 = \frac{5}{6}cx$ for $x > 2a$, and is not continuous. Hence the error y_2 is given by $y_2 = 0$ for $0 \leqq x < a$, $y_2 = -\frac{1}{6}ca + \frac{5}{6}c(x - a)$ for $a < x < 2a$, and $y_2 = -\frac{1}{3}ca + \frac{1}{6}c(x - 2a)$ for $x > 2a$, and, except for $x = a$ and $x = 2a$ satisfies the differential equation

$$Dy_2 = (1 - DS)f(x) = \begin{cases} 0, & 0 \leqq x < a, \\ \frac{1}{6}c, & a < x < 2a, \\ \frac{1}{3}c, & x > 2a, \end{cases} \tag{10}$$

where the absolute value of the right member does not exceed $\frac{5}{6}c$.

Now the right member of (10) may be expressed as the sum of two step functions, namely, $f_1(x)=0$ for $0\leqq x<a$ and $f_1(x)=\frac{5}{6}c$ for $x>a$; $f_2(x)=0$ for $0\leqq x<2a$ and $f_2(x)=(\frac{1}{6}-\frac{5}{6})c$ for $x>2a$. For these functions we have, using (10),

$$(1-DS)f_1(x)=\begin{cases}0, & 0\leqq x<a,\\ \frac{5}{6}(\frac{5}{6})c, & a<x<2a,\\ \frac{1}{6}(\frac{5}{6})c, & x>2a;\end{cases}$$

$$(1-DS)f_2(x)=\begin{cases}0, & 0\leqq x<2a,\\ \frac{5}{6}(\frac{1}{6}-\frac{5}{6})c, & 2a<x<4a,\\ \frac{1}{6}(\frac{1}{6}-\frac{5}{6})c, & x>4a.\end{cases}$$

Thus, the second generation error y_4, except for $x=a$, $x=2a$ and $x=4a$, satisfies the differential equation,

$$Dy_4=(1-DS)^2f(x)=(1-DS)f_1(x)+(1-DS)f_1(x)$$

$$=\begin{cases}0, & 0\leqq x<a,\\ (\frac{5}{6})^2c, & a<x<2a,\\ \frac{5}{6}[\frac{1}{6}c+\frac{5}{6}(\frac{1}{6}-\frac{5}{6})c], & 2a<x<4a,\\ \frac{5}{6}[\frac{1}{6}c+\frac{1}{6}(\frac{1}{6}-\frac{5}{6})c], & x>4a,\end{cases}$$

where now the absolute value of the right member does not exceed $(\frac{5}{6})^2c$.

Similarly, it may be shown by mathematical induction that $y_{2j}(j=2,3,\cdots)$, except for discontinuities at $x=a$, $x=2a$, $x=4a$, $\cdots$, $x=2ja$, satisfies a differential equation of the form (2) wherein the absolute value of the right member does not exceed $(\frac{5}{6})^jc$. Clearly, then, the continuous parts of y_{2j} satisfy the condition $\lim_{j\to\infty}y_{2j}=0$ (uniformly in every finite interval $0\leqq x\leqq X$) and the integral of $f(x)$ is given by the continuous part of (8).

GREATEST COMMON DIVISOR OF SEVERAL INTEGERS AND AN ASSOCIATED LINEAR DIOPHANTINE EQUATION

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1. Introduction. This note calls attention to a procedure—apparently overlooked in the past—for computing the greatest common divisor of a set of n integers and for solving, at the same time, a linear diophantine equation in which these n integers appear as coefficients. The procedure stems from the constructive aspect of a standard proof of the well-known [1]

THEOREM. *For given positive integers* $a_1,\cdots,a_n$, *the least positive number of the form* $a_1x_1+\cdots+a_nx_n$ ($x_1\cdots,x_n$ *integers*) *is* $d=(a_1\cdots,a_n)$, *the greatest common divisor of* $a_1,\cdots,a_n$.

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The proof involves showing that if integers $x_1, \cdots, x_n$ are chosen so that $0<a_1x_1+\cdots+a_nx_n=s\neq d$, then there exist integers $x_1', \cdots, x_n'$ such that $0<a_1x_1'+\cdots+a_nx_n'=r<s$. This may be accomplished as follows: Since $s\neq d$, and since d necessarily divides s, there is at least one of the a_j—which, for notational simplicity, we take to be a_1—such that $a_1=qs+r$ with $0<r<s$; it then follows that

$$a_1 = q(a_1x_1 + a_2x_2 + \cdots + a_nx_n) + r, \tag{1}$$

whence

$$0 < a_1(1 - qx_1) + a_2(-qx_2) + \cdots + a_n(-qx_n) = r < s. \tag{2}$$

Thus, with $x_1' = 1-qx_1$, $x_j' = -qx_j$ $(j=2, \cdots, n)$, the assertion is proved.

2. Greatest common divisor. The proposed procedure for computing $d=(a_1, \cdots, a_n)$ is the following: With $x_1^{(0)}, \cdots, x_n^{(0)}$ chosen arbitrarily, we use the technique of the above proof to determine $x_1^{(1)}, \cdots, x_n^{(1)}$ so that

$$r_1 = a_1x_1^{(1)} + \cdots + a_nx_n^{(1)} < a_1x_1^{(0)} + \cdots + a_nx_n^{(0)} = r_0 \neq d;$$

repetition of the process—in which we proceed from the $x_j^{(k)}$ and r_k to the $x_j^{(k+1)}$ and r_{k+1}—is carried out until finally we achieve an r_t that divides all of $a_1, \cdots, a_n$; then, clearly, $r_t=d$.

From the proof we note that each r_k $(k\geqq 1)$ is obtained directly as the least positive remainder on division of one of the a_j by r_{k-1}, so that in our quest for d there is no need actually to compute the $x_j^{(k)}$ for $k\geqq 1$. (See Sec. 3 below, however.) If, in carrying out the procedure, we find that a given r_{k-1} divides a_1, say, then r_k must be obtained through division of one of the other a_j by r_{k-1}; if, however, r_{k-1} divides *all* the a_j, then $r_{k-1}=d$.

The process may be shortened (*i.e.*, is not lengthened) through the use of least absolute remainders for the successive r_k (as in the example worked out in Sec. 4 below) and through a reasonably sensible choice of the arbitrary $x_1^{(0)}, \cdots, x_n^{(0)}$. Only slight modification of the analysis is required to justify the use of least absolute remainders: Indeed, if in (1) we have $0<-r<s$, we obtain in place of (2)

$$0 < a_1(qx_1 - 1) + a_2(qx_2) + \cdots + a_n(qx_n) = -r < s. \tag{3}$$

3. Linear diophantine equation. If in the computation of d delineated above we also compute the quantities $x_j^{(k)}$, we eventually arrive at a solution of the equation

$$a_1x_1 + \cdots + a_nx_n = d. \tag{4}$$

Indeed, if $r_t=d$, then (4) is solved by $x_1=x_1^{(t)}, \cdots, x_n=x_n^{(t)}$.

Confronted with the trivially more general problem of solving

$$a_1y_1 + \cdots + a_ny_n = N, \tag{5}$$

we proceed first to determine $d=(a_1, \cdots, a_n)$. If it turns out that d does not divide N, we know that (5) has no solution. If, however, d divides N, we next proceed to compute a set of $x_1, \cdots, x_n$ satisfying (4); then, clearly, $y_j=(N/d)x_j$ $(j=1, \cdots, n)$ provides a solution of (5).

The rule for computing the successive $x_j^{(k)}$ is embodied in (2) if, for the moment, we employ least positive remainders only:

$$x_j^{(k+1)} = -q_k x_j^{(k)} \ (j \neq p), \qquad x_p^{(k+1)} = 1 - q_k x_p^{(k)}, \tag{6}$$

where $a_p=q_k r_k+r_{k+1}$ $(0<r_{k+1}<r_k)$ and a_p is any one of the coefficients not divisible by r_k.

Introducing the Kronecker delta ($\delta_{jp}=0$ if $j\neq p$, $\delta_{pp}=1$) and using (3) to take advantage of least absolute remainders, we revise (6) to obtain for the successive computations of the $x_j^{(k)}$:

$$x_j^{(k+1)} = -(r_{k+1}/|r_{k+1}|)(q_k x_j^{(k)} - \delta_{jp_k}) \qquad (j=1, \cdots, n), \tag{7}$$

where $a_{p_k}=q_k|r_k|+r_{k+1}$ $(0<|r_{k+1}|\leqq \frac{1}{2}|r_k|)$. To give the procedure a unique determination (once the $x_j^{(0)}$ are arbitrarily assigned for $j=1, \cdots, n$), we may (i) provide that if $|r_{k+1}|=\frac{1}{2}|r_k|$, we choose $r_{k+1}>0$, and (ii) let p_k be the least integer p such that a_p is not divisible by r_k.

If our problem is merely to solve the diophantine equation (5), a possible short cut may present itself in the fortuitous result $|r_u|=N$ for some $u\geqq 0$. In this event we have $y_j=x_j^{(u)}$ $(j=1, \cdots, n)$ as a solution of (5).

4. Numerical example. We illustrate the method by obtaining a solution of the diophantine equation

$$1302y_1 + 1911y_2 + 5460y_3 + 15015y_4 = N, \tag{8}$$

where N is a given positive integer; at the same time we compute $d=(1302, 1911, 5460, 15015)$, which—as pointed out in Section 2 above—can be accomplished without consideration of (8).

In the solution table that follows, the figures in the final three columns are set down directly after the assignment of $x_j^{(0)}$ $(j=1, 2, 3, 4)$ and prior to computation of the $x_j^{(k)}$ for $k\geqq 1$; the reasons for proceeding so are implicit in the second and final paragraphs of Section 3 above.

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$	$x_4^{(k)}$	r_k	$-(r_{k+1}/\|r_{k+1}\|)q_k$	p_k
0	5	2	1	−1	777	2	1
1	9	4	2	−2	−252	−5	1
2	−44	−20	−10	10	42	−45	2
3	1980	901	450	−450	21		

Since $r_3 = 21$ divides each of 1302, 1911, 5460, and 15015, we have the result $(1302, 1911, 5460, 15015) = 21$. Also, a solution of (8)—provided N is a multiple of 21—is $y_1 = 1980(N/21)$, $y_2 = 901(N/21)$, $y_3 = 450(N/21)$, $y_4 = -450(N/21)$.

5. Remark. For $n \geq 4$, the above method for obtaining a solution of (5) in integers is considerably more direct than the method one finds in text books ([2], for example). The latter, however, has the advantage of leading to a general solution, while the method outlined in Section 3 provides only particular solutions. (Distinct choices of the sets of $x_j^{(0)}$ generally lead to distinct solutions.)

References

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2. J. V. Uspensky and M. A. Heaslet, Elementary Number Theory, New York, 1939, pp. 62–65.

A NOTE ON CONVERGENCE IN TOPOLOGICAL SPACES

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Let $\{x_n\}$ be an infinite sequence of points in a topological space S.

DEFINITION. *The sequence $\{x_n\}$ is said to be an $\mathfrak{D}$-sequence if and only if for every O open in S such that $x_n \in O$ for an infinite number of n, then $x_n \in O$ for $n \geq N$ for some integer N.*

Example 1. A convergent sequence is not necessarily an $\mathfrak{D}$-sequence. To show this let $x_n \equiv (-1)^n(1/n)$ in the space of reals. This is a convergent sequence, but if $O \equiv \{x \mid x < 0\}$, then $x_n \in O$ for an infinite number of n, but $x_n \notin O$ for an infinite number of n. Thus $\{x_n\}$ is not an $\mathfrak{D}$-sequence.

Example 2. An $\mathfrak{D}$-sequence is not necessarily convergent. Let S be the set of all positive integers; we topologize S by letting the open sets be $\varnothing$, S, and O_i where $O_i \equiv \{1, \cdots, i\}$. Let $x_n \equiv n$ for all n. Then $\{x_n\}$ is clearly an $\mathfrak{D}$-sequence which is not convergent. Note that this space is not a T_1-space.

THEOREM 1. *Let S be a T_1-space and $\{x_n\}$ an $\mathfrak{D}$-sequence in S. Then $\{x_n\}$ is a convergent sequence.*

Proof. Let $X \equiv \bigcup_n x_n$. Case 1: Suppose X is finite. Let $y = x_{n_1} = x_{n_2} = \cdots = x_{n_i} = \cdots$ where $\{x_{n_i}\}$ is a subsequence of $\{x_n\}$. Then $\lim x_n = y$; for if $y \in O$ where O is open in S, then $x_n \in O$ for $n \geq N$ since $\{x_n\}$ is an $\mathfrak{D}$-sequence. Case 2: Suppose X is infinite and $X' \neq \varnothing$, where X' is the derived set of X. Take $p \in X'$. Then $\lim x_n = p$; for if $p \in O$ is an open set, then $O \cap X$ is infinite since S is T_1 and thus $x_n \in O$ for $n \geq N$. Case 3: X is infinite and $X' = \varnothing$. We will show that this case cannot occur. Let $y_1, y_2, \cdots, y_i, \cdots$ be an infinite sequence of distinct points in X. Then $y_i \notin X'$ for all i. Then for each i, there exists an open set O_i such that $y_i \in O_i$ and $X \cap O_i = y_i$. Let $O^* = \bigcup_{i=1}^{\infty} O_{2i}$. Then $O^* \cap X$ is infinite and $x_n \in O^*$ for an infinite number of n, but $x_n \notin O^*$ for an infinite number of n since $y_{2i+1} \notin O^*$ for all i.

The converse of Theorem 1 is false as Example 1 shows.

THEOREM 2. *If S is compact and $\{x_n\}$ is an $\mathfrak{D}$-sequence, then $\{x_n\}$ is convergent.*

Proof. Suppose $\{x_n\}$ is not convergent. Then for $y \in S$, there exists an O_y open and containing y such that $x_n \in O_y$ for at most a finite number of n. Then $S = \bigcup_{y \in S} O_y$ and since S is compact, $S = \bigcup_{i=1}^{n} O_{y_i}$. It follows then that $x_n \in S$ for at most a finite number of n, a contradiction.

THEOREM 3. *If S is Hausdorff and $\{x_n\}$ an $\mathfrak{D}$-sequence, then $x_n \equiv x$ for $n \geqq N$ and for some $x \in S$.*

Proof. Since S is Hausdorff and hence T_1, by Theorem 1, $\{x_n\}$ is convergent. Let $\lim x_n = x$. We assert now that $x_n \equiv x$ for $n \geqq N$ for some N. Suppose on the contrary that $x_n \neq x$ for an infinite number of n. Let $X \equiv \bigcup_{x_n \neq x} x_n$. If X is finite, then $y = x_{n_1} = x_{n_2} = \cdots = x_{n_i} = \cdots$, where $x_{n_i} \in X$ and then $\lim x_n = y$. But in Hausdorff spaces limits are unique and thus $x = y$, a contradiction. We see then that X must be infinite. Let $x_{n_1}, x_{n_2}, \cdots, x_{n_j}, \cdots$ be an infinite sequence of distinct points in X. For each j we have $x_{n_j} \in O_j$, and $x \in O_j^{\#}$ where O_j and $O_j^{\#}$ are disjoint open sets in S. $O_j \cap X^*$ is finite for all j where $X^* \equiv \bigcup_{j=1}^{\infty} x_{n_j}$. Choose $O_j^* \subset O$ such that $O_j^* \cap X^* = x_{n_j}$; then let $O^* \equiv \bigcup_{j=1}^{\infty} O_{2j}^*$. Then $x_n \in O^*$ for an infinite number of n, but $x_n \notin O^*$ for an infinite number of n and thus $\{x_n\}$ is not an $\mathfrak{D}$-sequence, a contradiction.

THEOREM 4. *Let $f: S \to T$ be continuous, where S and T are topological spaces. Let $\{x_n\}$ be an $\mathfrak{D}$-sequence in S. Then $\{f(x_n)\}$ is an $\mathfrak{D}$-sequence in T.*

Proof. Suppose $f(x_n) \in O^*$ for an infinite number of n, where O^* is open in T. Then $x_n \in f^{-1}(O^*)$ for an infinite number of n and $f^{-1}(O^*)$ is open in S. Then $x_n \in f^{-1}(O^*)$ for $n \geqq N$ and hence $f(x_n) \in O^*$ for $n \geqq N$. Thus $\{f(x_n)\}$ is an $\mathfrak{D}$-sequence.

ON SOME EQUATIONS INVOLVING FUNCTIONS $\phi(n)$ AND $\sigma(n)$

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In this paper $\phi(n)$ denotes Euler's totient function, $\sigma(n)$, the sum of divisors of n, and F_n the Fermat number $2^{2^n}+1$.

I. The equation $\phi(x+k) = \phi(x) + \phi(k)$.

THEOREM. *If k is a positive integer and* (A) $k \not\equiv 3 \pmod 6$ *or* (B) $k = m \cdot \prod_{i=0}^{s} F_i^{\alpha_i}$, *where* $(m, 2\prod_{i=0}^{s+1} F_i) = 1$, $\alpha_i \geqq 2$ *and F_{s+1} is a prime number, then the equation*

$$\phi(x+k) = \phi(x) + \phi(k) \tag{1}$$

has at least one solution.

Proof. We distinguish three cases:

(a) $k \equiv 0$ or $\pm 2 \pmod 6$. Hence $k = 2^r n$, where $r \geqq 1$ and $(n, 2) = 1$. Then

$x=k$ is one of the solutions of equation (1). This follows from $\phi(k+k)=\phi(2^{r+1}n)=2^r\phi(n)$ and $\phi(k)+\phi(k)=2\phi(k)=2\phi(2^rn)=2^r\phi(n)$.

(b) $k\equiv\pm1 \pmod 6$. Hence $(k, 2)=(k, 3)=1$. Then $x=2k$ is a solution of equation (1), because the equalities $\phi(2k+k)=\phi(3k)=\phi(3)\phi(k)=2\phi(k)$ and $\phi(2k)+\phi(k)=\phi(k)+\phi(k)=2\phi(k)$ hold.

(c) The condition (B) is satisfied. It follows from the identity $2+\prod_{i=0}^{s}F_i=F_{s+1}$ that the equalities*

$$\phi(2m\prod F_i^{\alpha_i-1}+m\prod F_i^{\alpha_i})=\phi((m\prod F_i^{\alpha_i-1})(2+\prod F_i))$$
$$=\phi(m\prod F_i^{\alpha_i-1}\cdot F_{s+1})=\phi(m)\phi(\prod F_i^{\alpha_i-1})\cdot(F_{s+1}-1),$$
$$\phi(2m\prod F_i^{\alpha_i-1})+\phi(m\prod F_i^{\alpha_i})=\phi(2)\phi(m)\phi(\prod F_i^{\alpha_i-1})+\phi(m)\phi(\prod F_i^{\alpha_i})$$
$$=\phi(m)\phi(\prod F_i^{\alpha_i-1})+\phi(m)\cdot\prod F_i\cdot\phi(\prod F_i^{\alpha_i-1})$$
$$=\phi(m)\phi(\prod F_i^{\alpha_i-1})\cdot(1+\prod F_i)=\phi(m)\phi(\prod F_i^{\alpha_i-1})\cdot(F_{s+1}-1)$$

hold and hence that $x=2m\cdot\prod_{i=0}^{s}F_i^{\alpha_i-1}$ satisfies equation (1). Thus the theorem is proved.

$k=3$ is the least integer for which I do not know whether (1) is solvable. J. Browkin† proved that the equation $\phi(x+3)=\phi(x)+\phi(3)$ has no solution in integers $x<37$, 182, 142. Also I do not know whether (1) is solvable for $k=18t\pm3$, where $t=1, 2, \cdots$ and for $k=45$, 135, 315, 405, 495, 585, 765, 855, 945. For other integers $k\leqq1000$ the equation (1) is solvable.

Let $a(n)$ denote the number of integers $k\leqq n$ for which (1) is not solvable. Since such numbers may occur only in three arithmetical progressions: $18t\pm3$, $90t+45$, we have

$$\limsup\frac{a(n)}{n}\leqq\frac{1}{18}+\frac{1}{18}+\frac{1}{90}=\frac{11}{90}.$$

II. Equations involving the functions $\sigma((n)$.

In [4] L. Moser gave some results concerning the equations $\phi(x)=\phi(x+1)$ and $\phi(x+2)=\phi(x)+2$.

Below I give similar results concerning the equations

$$\sigma(x)=\sigma(x+1), \tag{2}$$
$$\sigma(x+2)=\sigma(x)+2. \tag{3}$$

1. The equation (2) has only nine solutions in positive integers $x\leqq10{,}000$; namely $x=14$, 206, 957, 1334, 1364, 1634, 2685, 2974, and 4364.

2. The equation (3) is satisfied by integers x such that both x and $x+2$ are primes. This equation also has three solutions in integers $x\leqq9998$ where x is composite; namely $x=434$, 8575, and 8825.

* For brevity we write $\prod$ instead of $\prod_{i=0}^{s}$.

† In a private communication.

Verification of these facts follows from Glaisher's tables [2].

I do not know whether equations (3) and the second equation of Moser are simultaneously solvable in composite integers.

III. On the generalization amicable numbers.

L. E. Dickson [1] and Th. E. Mason [3] defined amicable k-tuples as sets of k positive integers $n_1, \cdots, n_k$ such that $\sigma(n_1) = \sigma(n_2) = \cdots = \sigma(n_k) = n_1+n_2+\cdots+n_k$ and they gave some numerical examples of amicable triples. I examined the Glaisher tables [2] and found the following amicable triples:

(a) $2^2\cdot3^2\cdot5\cdot11$, $2^5\cdot3^2\cdot7$, $2^2\cdot3^2\cdot71$

(b) $2^3\cdot3\cdot5\cdot31$, $2^2\cdot3\cdot5\cdot29$, $2^2\cdot3\cdot5\cdot29$.

There exist pairs of "almost amicable numbers" satisfying the equations $\sigma(m) = \sigma(n) = m+n+1$, *e.g.* $\langle m, n\rangle = \langle 48, 75\rangle$, $\langle 140, 195\rangle$, $\langle 1050, 1925\rangle$.

References

1. L. E. Dickson, Amicable number triples, this MONTHLY, vol. 20, 1913, pp. 84–92.
2. J. W. L. Glaisher, Number-divisor Tables. Cambridge 1940.
3. Th. E. Mason, On amicable numbers and their generalizations, this MONTHLY, vol. 28, 1921, pp. 195–200.
4. L. Moser, Some equations involving Euler's totient function, this MONTHLY, vol. 56, 1949, pp. 22–23.

A THEOREM ON FINITE ABELIAN GROUPS

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Let G_m be the multiplicative group of integers less than and relatively prime to m, with multiplication defined modulo m.

THEOREM. *Given a finite Abelian group* $\mathfrak{G}$, *there exists an integer* m *such that* $\mathfrak{G}$ *is isomorphic to a subgroup of* G_m.

Proof. By the fundamental theorem of Abelian groups, $\mathfrak{G}$ is isomorphic to the direct product of cyclic groups of prime power order; *i.e.*, $\mathfrak{G}\cong\mathfrak{G}_1\otimes\cdots\otimes\mathfrak{G}_n$ where the order of $\mathfrak{G}_i$ is $p_i^{a_i}$.

From Dirichlet's theorem on the infinitude of primes in an arithmetic series, it follows that n distinct primes q_i can be found such that $q_i = a_i p_i^{a_i}+1$. As there exist primitive roots for every prime, there is an element y_i in G_{q_i} such that y_i generates G_{q_i}. Let $x_i = y_i^{a_i}$. Then x_i generates a cyclic subgroup H_i of G_{q_i} whose order is $p_i^{a_i}$.

Let $\mathfrak{F} = G_{q_1}\otimes\cdots\otimes G_{q_n}$ and $\mathfrak{H} = H_1\otimes\cdots\otimes H_n$. $\mathfrak{H}$ is isomorphic to $\mathfrak{G}$. By a known theorem,* $\mathfrak{F}$ is isomorphic to G_m with $m = \prod_{i=1}^{n} q_i$. Thus the theorem is proved.

* H. S. Vandiver and Milo W. Weaver, Introduction to Arithmetic Factorization and Congruences from the Standpoint of Abstract Algebra, this MONTHLY, vol. 65, October, Part II, 1958, pp. 39–41.

CORRECTIONS TO "VOLUMES AND AREAS OF CROSS-SECTIONS"*

HERBERT BUSEMANN, University of Southern California

In the second line from the bottom on page 249, the factor cos u is missing. Therefore $e^{2|u|}$ and $e^{-2|u|}$ in the definition of $F(u, h)$ must be replaced by $e^{t|u|}$ and $e^{-t|u|}$ with a suitable $t>2$ to attain

$$(4\pi)^{-1}V'(B_0) = \int_0^{1/2\pi} e^{tu}g(u)du < 0, \qquad g(u) = (\cos 2u + \tfrac{1}{2})\cos u.$$

The inequalities $g(u) \leqq b = 3/2$, $g(u) \leqq 0$ in $[\frac{1}{3}\pi, \frac{1}{2}\pi]$, $g(u) \leqq -c < 0$ in $[\alpha, \beta]$ if $\frac{1}{3}\pi < \alpha < \beta < \frac{1}{2}\pi$ give

$$(4\pi)^{-1}V'(B_0) < b\int_0^{\alpha} e^{tu}du - c\int_{\alpha}^{\beta} e^{tu}du < t^{-1}e^{\beta t}[(b+c)e^{(\alpha-\beta)t} - c],$$

which tends to $-\infty$ for $t\to\infty$.

* This MONTHLY, vol. 67, 1960, pp. 248–250.

CLASSROOM NOTES

EDITED BY C. O. OAKLEY, Haverford College

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A CLASSROOM PROOF OF $\lim_{t\to 0}(\sin t)/t = 1$

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The usual proof of $\lim_{t\to 0}(\sin t)/t = 1$ requires that the inequalities: $\sin t < t < \tan t$ be established for sufficiently small (positive) values of t. The latter part of this string of inequalities is usually determined by comparing the area of a unit-circular sector of angle t with the area of a triangle. If the finding of the area of a circular sector exhausts the students' reservoir of thinking, then an understanding of the remainder of the proof is lost. The following method of proof avoids the use of area.

Let it already have been established (either by definition or by some kind of geometric-trigonometric methods) that the point with coordinates $(\cos t, \sin t)$ is at a distance, measured appropriately along the circumference of the unit circle, of $|t|$ units from the point $(1, 0)$.

THEOREM 1.

$$\lim_{t\to 0}(1 - \cos t)/t = 0.$$

Proof. Since the straight-line distance between $(\cos t, \sin t)$ and $(1, 0)$ does

not exceed the arc length along the circle between the same two points, we have

$$0 \leqq \sqrt{\{(\cos t - 1)^2 + (\sin t - 0)^2\}} \leqq |t| .$$

Squaring, simplifying, and dividing by $2t$ yields

$$0 \leqq (1 - \cos t)/t \leqq \tfrac{1}{2}t, \qquad t > 0; \qquad \tfrac{1}{2}t \leqq (1 - \cos t)/t \leqq 0, \qquad t < 0.$$

By the "pinching theorem" for limits, the theorem is proved.

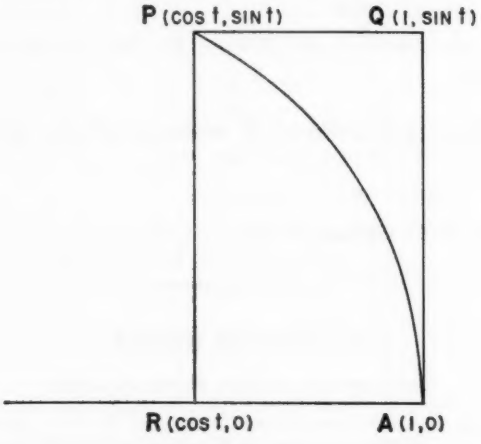


FIG. 1

THEOREM 2.

$$\lim_{t \to 0} (\sin t)/t = 1.$$

*Proof.** If $0 < t < \frac{1}{2}\pi$, let P be the point $(\cos t, \sin t)$, Q, the point $(1, \sin t)$, A, the point $(1, 0)$, and R, the point $(\cos t, 0)$. (Fig. 1) Then $\overline{PR} \leqq \text{arc } PA \leqq \overline{PQ} + \overline{QA}$ and therefore

$$\sin t \leqq t \leqq 1 - \cos t + \sin t.$$

But $\sin t < t$ implies $1 - \cos t + \sin t < 1 - \cos t + t$ so that

$$t \leqq 1 - \cos t + \sin t \leqq 1 - \cos t + t.$$

Hence, dividing by t,

$$1 \leqq \frac{1 - \cos t}{t} + \frac{\sin t}{t} \leqq \frac{1 - \cos t}{t} + 1.$$

By Theorem 1 and the "pinching theorem" for limits, the theorem is proved.

* For $t \to 0+$, the proof for $t \to 0-$ being similar.

DERIVATIVES OF sin θ AND cos θ

C. S. OGILVY, Hamilton College

The derivatives of the sine and cosine functions may be presented handily with the aid of vectors, following a suggestion of R. T. Coffman of Richland, Washington.

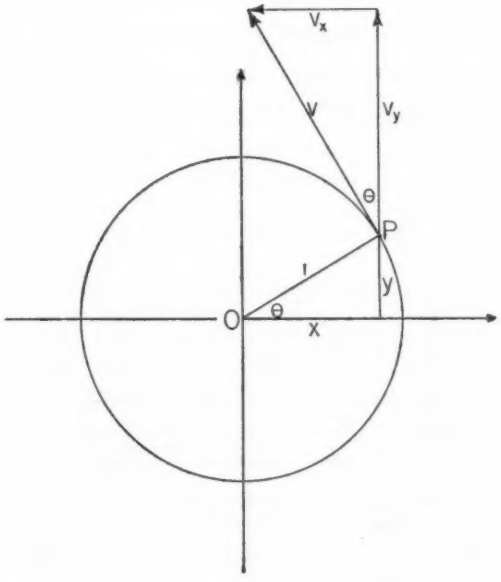


FIG. 1

Consider the point $P(x, y)$ moving around the unit circle (Fig. 1) in the direction of increasing θ; then $x=\cos\theta$ and $y=\sin\theta$. We note that $ds/d\theta=1$ if θ is measured in radians. The direction of motion of P is tangential to the circle; hence the vector v is perpendicular to OP at P. From the vector triangle,

$$\sin\theta=\frac{|v_x|}{|v|}=\left|\frac{dx}{dt}\,\frac{dt}{ds}\right|\frac{ds}{d\theta}=\left|\frac{dx}{d\theta}\right|,$$

and we must make it minus because it is in the direction of decreasing x. Exactly similarly, $dy/d\theta=\cos\theta$.

EQUIANGULAR COMPLEX NUMBERS

J. P. BALLANTINE, University of Washington

1. Introduction. Consider the complex number,*

$$z = 0.959913 \text{ cis } 1.440881 \text{ radians.} \tag{1}$$

Rough calculation shows that

$$z^4 = 0.849039 \text{ cis } (-0.5196619 + 2\pi),$$
$$z^{13} = 0.587509 \text{ cis } (-0.1181050 + 6\pi).$$

The numbers z^0, z^4, and z^{13} plotted in the complex plane are the vertices of a nearly equilateral triangle (Fig. 1). The same holds for z^k, z^{k+4}, and z^{k+13}.

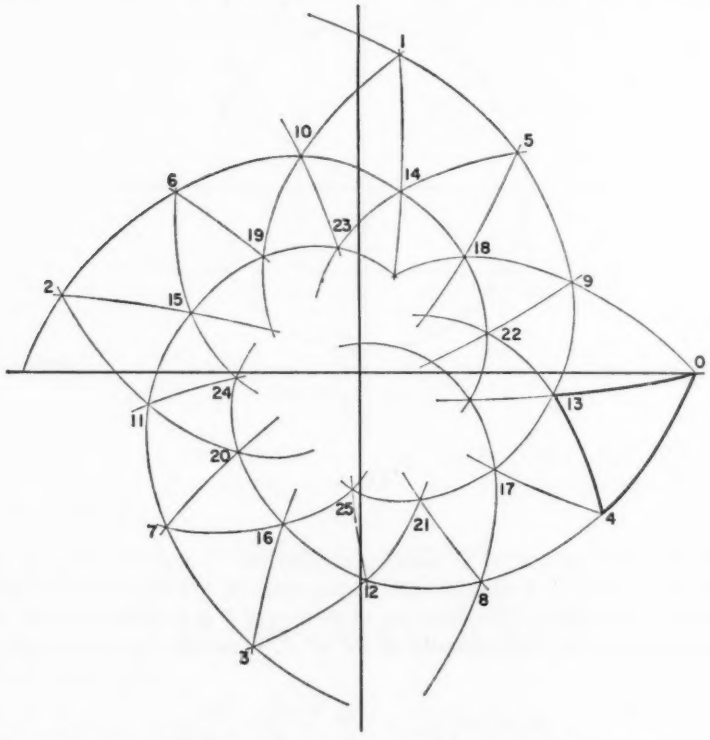


FIG. 1. Graphs of the values of z^n for integral values of n, where z is the equiangular complex number for $p=4$, $q=13$.

More generally, suppose that

* Cis θ is a common abbreviation for $\cos\theta + i \sin\theta$.

$$z = r \operatorname{cis} \theta, \tag{2}$$

where r and θ are fixed and $|r| \neq 0$ or 1. Let n assume the values $n_1, n_2, \cdots$, of an integral-valued arithmetic progression. Then the complex numbers z^n fall along a spiral whose formula in the polar coordinates ρ, ϕ is of the form

$$\rho = Re^{h\phi}. \tag{3}$$

The values of R and h can be determined by substituting z^n for $n = n_1$ and n_2 into (3). The formula for h is

$$h = \frac{(n_2 - n_1) \log r}{\theta(n_2 - n_1) + 2\pi(m_2 - m_1)}.$$

R is readily found from h. Because of the ambiguity in $(m_2 - m_1)$, many different value of h are possible, giving rise to many different possible spirals. For simplicity, we choose $m_2 - m_1$ so that $|h|$ is as large as possible. Then there is only one spiral, except in uninteresting cases where there may be two. So z and any integral-valued arithmetic progression of two or more terms determines one spiral.

A spiral triangle is determined by its three vertices. For sides it has the spirals determined by taking its vertices two at a time. If it should happen that the three numbers z^0, z^4, and z^{13} determine a spiral triangle that is equiangular, then the spiral triangle z^0, z^9, and z^{13} also is equiangular, and the configuration showing all the integral powers of z comprises an infinitude of equiangular spiral triangles covering the entire complex plane except the points 0 and ∞. This would be an idealization of designs frequently found in nature.

DEFINITION. *If p and q are any assigned relatively prime positive integers, and if z^0, z^p, and z^q are the vertices of an equiangular spiral triangle, then z is called an equiangular complex number for p and q.*

If z is equiangular for p and q, then so are z^{-1}, $\bar{z}$ the conjugate of z, and $\bar{z}^{-1}$. The values $\theta = 0$ or π and $r = 0$ or 1 are impossible. (The three cube roots of unity determine an equiangular spiral triangle whose angles are π, which we exclude, since we demand that the angles of an equiangular spiral triangle be $\pi/3$.) One of the four numbers, z, $\bar{z}$, z^{-1}, $\bar{z}^{-1}$ will fall in the upper half of the unit circle, and that one we take as z.

2. The logarithm transformation. Let z of (2) be transformed into Z by

$$Z = \log z = \log r + i(\theta - 2\pi m). \tag{4}$$

Let

$$Z_p = \log z^p = p \log r + i(p\theta - 2\pi p'),$$
$$Z_q = \log z^q = q \log r + i(q\theta - 2\pi q').$$

The values of p' and q' are chosen so as to make the imaginary terms in Z_p and

Z_q small. Let

$$\Delta = \begin{vmatrix} p & q \\ p' & q' \end{vmatrix}.$$

The Z-plane is bounded by the lines $|Y| = \pi$. By our choice of p' and q', the transform of the spiral triangle O, Z_p, Z_q will be so small that it will not be broken by said bounding lines, and will be an ordinary equilateral triangle, since the transformation is conformal.

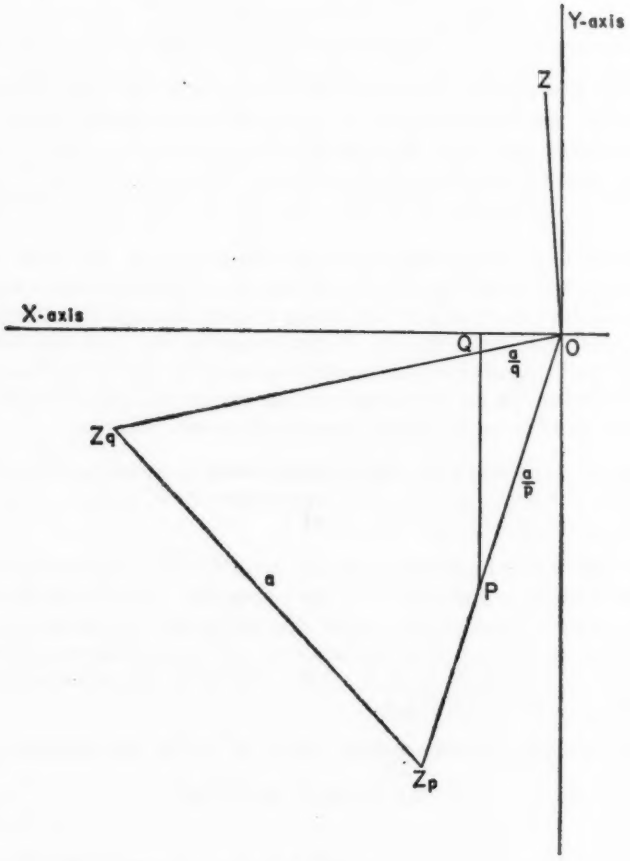


FIG. 2*

* To save space, the figure is purposely drawn out of scale. The line OZ should be many times the length shown and P, Q should be much nearer to O.

3. Formulas for finding equiangular complex numbers. Figure 2 shows the Z-plane, with origin at O. It shows the points Z, and O, Z_p, and Z_q, forming an equilateral triangle of side a. Also shown are the construction points, P and Q. P is on line OZ_p at a distance a/p from O. Q is on OZ_q at a distance a/q from O.

TABLE OF XY-COORDINATES OF FIVE POINTS

POINT	X	Y
Z	$\log r$	θ
Z_p	$p \log r$	$p\theta - 2\pi p'$
Z_q	$q \log r$	$q\theta - 2\pi q'$
P	$\log r$	$\theta - 2\pi p' \div p$
Q	$\log r$	$\theta - 2\pi q' \div q$

The value of a is found by equating two different expressions for the distance PQ. By the two point distance formula,

$$PQ = \left|2\pi\left(\frac{q'}{q} - \frac{p'}{p}\right)\right| = \frac{2\pi}{pq}\,|\,\Delta\,|\,.$$

PQ occurs in the triangle OPQ, $\angle POQ = 60°$, $OP = a/p$ and $OP = a/q$. By the law of cosines, $PQ = (aS)/(pq)$, where $S^2 = p^2 - pq + q^2$. $a = (2\pi/S)|\Delta|$. Now, applying the law of sines to the triangle OPQ,

$$\sin P = (p \sin 60°)/S, \qquad \cos P = (2q - p)/(2S).$$

The line PQ is parallel to the Y-axis. So the X- and Y- coordinates of Z_p can each be found by two different methods, and the results equated to give $\log r$ and θ.

$$X_p = p \log r = a \sin P,$$

$$Y_p = p\theta - 2\pi p' = a \cos P,$$

$$\log r = \frac{a \sin P}{p} = \frac{\sqrt{3}\pi\Delta^*}{S^2}\,.$$

$$\theta = \frac{1}{p}(2\pi p' + a \cos P) = \frac{1}{p}\left(2\pi p' + \frac{\pi\Delta(2q - p)}{S^2}\right).$$

4. Suggested procedure. Given p and q, the value of $z = r \operatorname{cis} \theta$ may be found by the following steps. Select p' and q', and compute Δ. The sign of Δ can be changed by interchanging p and q. The resulting z will be outside or within the unit circle according as Δ is plus or minus. Find S^2, $J = (\pi\Delta)/S^2$, $\log r = J\sqrt{3}$,

$$\theta = \frac{1}{p}(2\pi p' + (2q - p)J).$$

* Without the ambiguous sign, since $\sqrt{3}$ comes from sin 60°.

Example. Let $p=4$, $q=13$, take $p'=1$ and $q'=3$. Then $\Delta=-1$ and $S^2=133$;

$$J = -\pi \div 133 = -0.0236210,$$

$$\log r = J\sqrt{3} = -0.0409128,$$

$$\theta = \tfrac{1}{4}(2\pi + 22J) = \tfrac{1}{2}(\pi + 11J) = 1.4408808.$$

The resulting value of z appeared in (1).

Note that the choice $\theta=(2\pi-22J)\div 4$ does not lead to an equiangular complex number.

SOME SAMPLE EQUIANGULAR COMPLEX NUMBERS

p	q	p'	q'	Δ	r	θ
4	13	1	3	−1	0.95991286	1.4408808
8	21	3	8	1	0.98398308	2.3958140
21	73	2	7	1	0.99871657	0.6028121
19	103	7	38	1	0.99939645	2.3182883
19	103	2	11	3	0.99819046	0.6716797

5. Conclusion. A routine is to be designed for a digital computer to find complex roots by Newton's method. The first approximation is found by trying several scattered complex numbers within the unit circle. One easy way to tell the machine which numbers to try is to feed in one complex number z and direct the machine to use successive integral powers of z. For best results, z should be an equiangular complex number.

Nature comes to the same conclusion in arranging the points on a pine cone, or the inner blossoms of a daisy.

ON EULER'S EQUATION

HUGH A. THURSTON, University of British Columbia

I. Destructive

1. If u is a function of x and y, the equation $x(\partial u/\partial x)+y(\partial u/\partial y)=0$ is known as Euler's equation. In many books (*e.g.*, on p. 109 of Courant's *Differential and Integral Calculus*, vol. II) it is said to be a sufficient and (tacitly assuming u to be well enough behaved) necessary condition for the function to be homogeneous of degree 0. This is not exactly true.

2. A common "proof" is as follows. Let u be $\phi(x, y)$; and for a given x and y define ψ by

(i) $$\psi(t) = \phi(x, y) - \phi(x\cdot t, y\cdot t) \text{ for every } t.$$

Then (assuming that ϕ obeys the chain rule)

$$\psi'(t) = -x\cdot\phi_1(x\cdot t, y\cdot t) - y\cdot\phi_2(x\cdot t, y\cdot t) \text{ for every } t.$$

Thus $t\cdot\psi'(t)=0$, by Euler's equation for ϕ; hence, if $t\neq 0$, $\psi'(t)=0$.

Courant omitted the proviso "if $t\neq 0$," and deduced that ψ is a constant, whence $\psi(t)=\psi(1)$ for every t, *i.e.*, $\phi(x, y)=\phi(x\cdot t, y\cdot t)$ for every t.

3. In fact, the best that we can deduce without knowing more about $\psi'(0)$ is that $\psi(t)=\psi(1)$ when $t>0$, and $\psi(t)=\psi(-1)$ when $t<0$. Each of these shows that $\phi(x, y)=\phi(x\cdot t, y\cdot t)$ whenever $t>0$: *i.e.*, ϕ is positively-homogeneous of degree zero.

4. A positively-homogeneous solution of Euler's equation is

(ii) $\phi(x, y)=\xi(x/y)$ whenever $y>0$, $\eta(x/y)$ whenever $y<0$, a whenever $y=0$, for some differentiable functions ξ, η, and some number a. ϕ is not homogeneous unless $\xi=\eta$.

5. As a particular solution, I shall choose

$$\text{(iii)}\qquad \xi(z)=z/(e^z+e^{-z}),\qquad \eta(z)=2z/(e^z+e^{-z}),\qquad a=0.$$

It is easy to verify that $\phi_1(x, y)$ and $\phi_2(x, y)$ exist and Euler's equation is satisfied for every x and every nonzero y.

Now let us suppose that $y=0$. Firstly, $\phi_1(x, 0)=0$. Secondly,

$$\lim_{h\to 0^+}\frac{\phi(x, h)-\phi(x, 0)}{h}=\lim_{h\to 0^+}\frac{\xi(x/h)-a}{h}=\lim_{h\to 0^+}\frac{x}{h^2(e^{x/h}+e^{-x/h})}=0.$$

Similarly, $\lim_{h\to 0^-}[\phi(x, h)-\phi(x, 0)]/h=0$. Then $\phi_2(x, 0)$ exists (and is zero) for every x. Then $x\cdot\phi_1(x, 0)+0\cdot\phi_2(x, 0)=0$, and so the equation is satisfied always.

6. Why bother to verify that $\phi_2(x, 0)$ exists? Because although $0\cdot\phi_2(x, 0)$ cannot equal anything other than 0, it cannot truly equal zero unless it exists. From the practical aspect, too, if we were trying to solve the equation with a ϕ such that $\phi_2(x, y)$ fluctuated more wildly than y^{-1} for y near to 0, we could scarcely trust the mere "$x\cdot\phi_1(x, 0)=0$".

Equations (iii) were chosen with this in mind; (ii) and (iii) make $\phi_2(x, 0)$ exist for every x. If we relaxed this requirement, any differentiable ξ and η would make (ii) a solution.

7. It is perhaps unusual to think about what is really meant by "solve the equation $x\cdot\phi_1(x, y)+y\cdot\phi_2(x, y)=0$." It is (very naturally) more usual to concentrate on the techniques of solution. Nevertheless, let us continue. All, or nearly all, methods of solving partial differential equations use the chain rule. This rule is not universally valid; the usual condition under which it is proved is differentiability, which is much stronger than the mere existence of $\phi_1(x, y)$ and $\phi_2(x, y)$.

8. Now suppose that we solve the equation using the chain rule. Then we are unlikely to have the most general solution. But we might, if we are lucky, have the most general differentiable solution. In practice, this is likely to be general enough.

However, we use differentiability only to justify the chain rule. Why not, then, simply look for all solutions which obey this rule? This is more general than looking for differentiable solutions. We can go still further. We use only a special case of the chain rule. Let us then define ϕ to be *amenable on* R if

$$(d/dt)\phi(x\cdot t,\ y\cdot t) = x\cdot\phi_1(x\cdot t,\ y\cdot t) + y\cdot\phi_2(x\cdot t,\ y\cdot t)$$

whenever $(x\cdot t,\ y\cdot t)\in R$. If R is the set of all pairs of numbers, ϕ is simply *amenable*.

9. We can now find the most general amenable solution of Euler's equation. Returning to the proof in Section 3, we have $\psi'(t)=0$ whenever $t\neq 0$. $\psi'(0)$ exists (it equals $-x\cdot\phi_1(0,\ 0)-y\cdot\phi_2(0,\ 0)$). Then $\psi'(0)=0$, and so ψ really is constant. Then $\psi(1)=\psi(0)$, *i.e.*, $\phi(x,\ y)=\phi(0,\ 0)$. Thus ϕ is constant. Because a constant is differentiable, this is also the most general differentiable solution.

10. Thus neither strictly nor loosely is it true to say that the general solution of Euler's equation is a homogeneous function of degree zero. Strictly, the general differentiable (or amenable) solution is a constant; loosely the general solution includes nonhomogeneous functions.

II. Constructive

11. If E_+ denotes $\{(x,\ y)\,|\,y>0\}$, any of the usual proofs (including that in Sec. 3) yield:

THEOREM. *If* $x\cdot\phi_1(x,\ y)+y\cdot\phi_2(x,\ y)=0$ *for every* $(x,\ y)$ *in* E_+, *and if* ϕ *is amenable on* E_+, *then there is a differentiable* ξ *such that* $\phi(x,\ y)=\xi(x/y)$ *for every* $(x,\ y)$ *in* E_+.

The theorem remains true if we replace E_+ by $\{(x,\ y)\,|\,y<0\}$. We fill the gap with the trivial

THEOREM. *If* $x\cdot\phi_1(x,\ y)+y\cdot\phi_2(x,\ y)=0$ *whenever* $y=0$, *then there is a number* a *such that* $\phi(x,\ 0)=a$ *for every* x.

Proof. $x\cdot\phi_1(x,\ 0)=0$ for every x. Therefore $\phi_1(x,\ 0)=0$ for every nonzero x. $\phi_1(0,\ 0)$ exists (by hypothesis). Therefore $\phi_1(x,\ 0)=0$ for every x.

12. Together, the theorems of Section 11 yield:

THEOREM. *If* $x\cdot\phi_1(x,\ y)+y\cdot\phi_2(x,\ y)=0$ *for every* x *and* y, *and if* ϕ *is amenable except perhaps on* $\{(x,\ 0)\}$, *then there are differentiable functions* ξ *and* η *such that*

$$\text{(iv)}\qquad \phi(x,\ y) = \begin{cases} \xi(x/y) & \text{whenever } y>0,\\ \eta(x/y) & \text{whenever } y<0,\\ \phi(0,\ 0) & \text{whenever } y=0. \end{cases}$$

13. Now what about the converse: *i.e.*, what conditions on ξ and η are needed

for (iv) actually to give a solution? It is easily seen that (iv) satisfies the equation when $y \neq 0$; and it satisfies the equation when $y = 0$, provided that $\phi_2(x, 0)$ exists. So what we need are necessary and sufficient conditions that $\phi_2(u, 0)$ exist for every u. These are found with little trouble to be

$$\text{(v)} \qquad \begin{aligned} \xi(0) &= \eta(0) = \phi(0, 0), \\ \lim_{x \to \infty} x \cdot (\xi(x) - \xi(0)) &= \lim_{x \to -\infty} x \cdot (\eta(x) - \eta(0)), \\ \lim_{x \to -\infty} x \cdot (\xi(x) - \xi(0)) &= \lim_{x \to \infty} x \cdot (\eta(x) - \eta(0)). \end{aligned}$$

14. Perhaps the most important case is where ϕ is amenable everywhere except at the origin. The solution here will be given by (iv) and (v) together with whatever other conditions are needed to make ϕ amenable on $\{(x, 0) \mid x \neq 0\}$. $(d/dt)\phi(x \cdot t, 0) = (d/dt)\phi(0, 0) = 0$. $x \cdot \phi_1(x, 0) + 0 \cdot \phi_2(x, 0) = 0$. Therefore no extra condition is required.

15. A slightly different problem is to find the most general ϕ, amenable except at the origin, for which

$$x \cdot \phi_1(x, y) + y \cdot \phi_2(x, y) = 0 \quad \text{whenever} \quad (x, y) \neq (0, 0).$$

We argue as in Section 11, except that now we no longer know that $\phi_1(0, 0)$ exists. We find, in place of (iv):

$$\phi(x, y) = \xi(x/y) \quad \text{if} \quad y > 0; \qquad \eta(x/y) \quad \text{if} \quad y < 0;$$

$$a \quad \text{if} \quad y = 0 \quad \text{and} \quad x > 0; \qquad b \quad \text{if} \quad y = 0 \quad \text{and} \quad x < 0.$$

(*i.e.*, ϕ is the general positively-homogeneous function of degree zero).

In place of (v) we get:

$$\begin{aligned} \lim_{x \to \infty} x \cdot (\xi(x) - a) &= \lim_{x \to -\infty} x \cdot (\eta(x) - a), \\ \lim_{x \to -\infty} x \cdot (\xi(x) - b) &= \lim_{x \to \infty} x \cdot (\eta(x) - b). \end{aligned}$$

16. This is about as much as one can expect without a radically different type of attack, and without producing solutions which would be little appreciated—for few of us would be happy with functions which did not obey the chain rule.

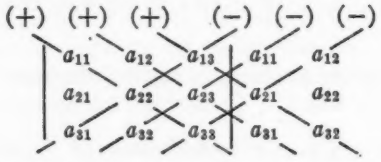
Of course, if ϕ is wanted only over some special region, we may get a simple answer. *E. g.*, if R is an open convex set, then ϕ is differentiable in R and $x \cdot \phi_1(x, y) + y \cdot \phi_2(x, y) = 0$ for every (x, y) in R if and only if ϕ is differentiable and homogeneous of degree zero on R (*i.e.*, $\phi(x, y) = \phi(x \cdot t, y \cdot t)$ whenever $t \neq 0$ and both $(x, y) \in R$ and $(x \cdot t, y \cdot t) \in R$).

CONCERNING FOURTH-ORDER DETERMINANTS

ROGER OSBORN, University of Texas

A problem which arises periodically in intermediate (and college) algebra courses is that of expanding fourth-order determinants. Many students have been indoctrinated in the diagrammatic method of expanding third-order determinants, and they wonder (silently mostly, but sometimes aloud) why the same type diagram cannot be used to expand a fourth-order determinant. Quite usually the student's question is answered merely by, "It can't be done." This note is devoted to a proof that it can't be done by a single diagram of the same type used for third-order determinants.

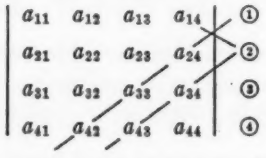
A diagrammatic scheme for expanding a third order determinant requires rewriting the first two columns and taking products down diagonals, some positive and some negative as shown in the diagram:



An obvious extension of this diagram for a fourth-order determinant would be to rewrite the first three columns and take diagonal products. This obviously does not yield the correct result since there are only eight such diagonal products instead of the twenty-four product terms given by the standard definition of the fourth-order determinant.

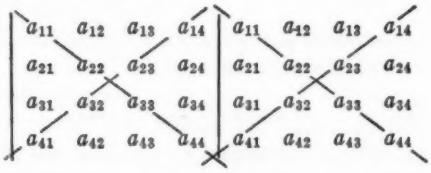
A question now arises. Can additional columns be added to the diagram, in sufficient number to give twenty-four diagonal products, and of such nature that the products will be the ones desired—*i.e.*, no two column or row indices will be alike in any single product and no two products will be alike?

The answer to this question is no. Suppose we append additional elements, in columns, to the diagram. At the outset we keep an open mind about which elements are to be appended. Add elements to the diagram as illustrated.



Added element ① must be a_{11} since its row and column subscripts may neither repeat a row or column subscript found on the diagonal shown terminating at ①. Element ② will lie on two diagonals, and since along these diagonals the only row subscript not already used is 2, and since the only column subscript not already used is 1, the element must be a_{21}. Similarly, element ③ must be a_{31}, and element ④ must be a_{41}.

Having determined the nature of the elements of the first added column (so that no diagonal product will contain any two elements from the same row or column), use the same device to determine the elements of the second added column. The elements of this column, from top to bottom, will be found to be the same as the elements of the second column of the given determinant, namely, $a_{12}, a_{22}, a_{32}, a_{42}$. Proceeding similarly, the next two added columns will be identical with the third and fourth columns, respectively, of the given determinant. As soon as the fourth column is repeated, though, it is observed that two sets of identical diagonal products have been created, violating the principle that no two products may be identical.



Hence, the diagonal scheme is found not to be valid for the expansion of fourth-order determinants since the additional elements which must be appended to avoid violating the principle that no product may contain two elements from the same row or column cause the other principle to be violated.

It might be of interest to note, before closing, that of thirty textbooks chosen at random, each of which included a discussion of determinants, sixteen (or more than half) gave the diagonal scheme for evaluating a third-order determinant. Of these sixteen, ten (or a third of the total number) did not even mention the impossibility of the use of the diagonal scheme for evaluating higher order determinants. Of the remaining six books, five stated that the diagonal scheme was invalid for $n>3$, but gave no justification for the statement. Only one book stated that the scheme was invalid for $n>3$ and gave a reason for the statement, and the only reason given was that the repetition of only three columns would not give a sufficient number of product terms. It is unfortunate that so many books give a scheme which cannot be generalized and then do not even mention that generalization is impossible.

ON A NOTE OF N. S. MENDELSOHN

K. VENKANNAYAH, Government College, Mercara, India

N. S. Mendelsohn in his note (this MONTHLY, vol. 58, 1951, p. 563) has proved the following results:

(1) *The sequence* $S_n=(1+1/n)^n$ *is a monotone increasing function of* n *for positive integral* n.

(2) *The sequence* $T_n=(1+1/n)^{n+1}$ *is a monotone decreasing function of* n *for positive integral* n.

The proofs of these results are based on the well-known inequality

$$(3) \qquad (a_1 + \cdots + a_n)/n > (a_1 \cdots a_n)^{1/n}.$$

We should note that Mendelsohn obtains (1) and (2) by applying (3) to the two sets of numbers

$$(4) \qquad \alpha_1 = 1, \quad \alpha_i = (n+1)/n, \quad i = 2, \cdots, n+1;$$

$$(5) \qquad \beta_1 = 1, \quad \beta_i = n/(n+1), \quad i = 2, \cdots, n+2.$$

Here we are interested in obtaining the results (1) and (2) by applying the inequality

$$(6) \qquad (p_1a_1 + \cdots + p_ka_k)/(p_1 + \cdots + p_k) > (a_1^{p_1} \cdots a_k^{p_k})^{1/(p_1+\cdots+p_k)}$$

where $p_1, \cdots, p_k$ are positive rationals, to either of the sets (4) and (5). In effect this amounts to applying (3) and

$$(7) \qquad (a_1 + \cdots + a_n)/n < (a_1^{a_1} \cdots a_n^{a_n})^{1/(a_1+\cdots+a_n)},$$

which emerge as particular cases of (6), to either of the sets (4) and (5).

Consider the set (4) and let $k=n+1$, $p_1 = \cdots = p_{n+1} = 1$, $a_1 = 1$, $a_2 = \cdots = a_{n+1} = (n+1)/n$ in (6). We get $S_{n+1} > S_n$, which proves (1). If, on the other hand, we set $k=n+1$, $p_1=1$, $p_2 = \cdots = p_{n+1} = (n+1)/n$, $a_1 = 1$, $a_2 = \cdots = a_{n+1} = n/(n+1)$ in (6), we obtain $T_{n+1} < T_n$, which proves (2).

If, however, we wish to consider (5) instead of (4), we can set in (6), $k=n+2$, $p_1 = \cdots = p_{n+2} = 1$, $a_1 = 1$, $a_2 = \cdots = a_{n+2} = n/(n+1)$ to get (2) and set $k=n+2$, $p_1=1$, $p_2 = \cdots = p_{n+2} = n/(n+1)$, $a_1 = 1$, $a_2 = \cdots = a_{n+2} = (n+1)/n$ to recover (1).

MATHEMATICAL EDUCATION NOTES

EDITED BY JOHN A. BROWN, University of Delaware, AND
JOHN R. MAYOR, AAAS and University of Maryland

All material for this department should be sent to John R. Mayor, 1515 Massachusetts Avenue, N.W., Washington 5, D. C.

QUALIFICATIONS AND TEACHING LOADS OF MATHEMATICS AND SCIENCE TEACHERS

KENNETH E. BROWN, U. S. Office of Education

Not long ago President Eisenhower asked citizens to scrutinize their schools to see whether they were meeting "the stern demands of the era we are entering." Many of the citizens following the President's suggestion began their scrutiny with teachers and pupils—for they are the heart of the school.

Because they realize that the school's success depends in large measure on the competence of the teachers, many citizens are looking closely at teachers—their preparation, experience, and work loads. And because they hear on every

side from leaders in government, education, business, and industry that to meet the stern demands of the era ahead men will need thorough training in mathematics and science, they are particularly interested in teachers of these subjects.

Visitors to some schools are encouraged by the information they get; visitors to others are discouraged.

To get a better picture of the mathematics and science teachers' preparation, the U. S. Office of Education in cooperation with New Jersey, Virginia, and Maryland, studied rather intensively the qualifications of teachers in these three states.* The purpose of the study was threefold: (1) To determine the background preparation of science and mathematics teachers in the selected states, (2) to determine the teaching load and the teaching combinations of these teachers, (3) to develop a technique for securing data which might serve as a pattern for other states in making their own studies.

In this study a mathematics or science teacher is defined as any teacher who taught one or more classes in one of these subjects in grades 9–12. There were 1393 such teachers in the study, of whom 662 taught mathematics, but no science, 594 taught science, but no mathematics, and 137 taught both mathematics and science.

The range in average semester hours of college courses per teacher was from 170 in one state to 143 in another state. The highest average number of semester hours (166) was for the teachers in the 3-year senior high schools and the lowest number (142) was in the 5-year junior-senior high schools. When one considers that 120 semester hours is the number required for the bachelor's degree in many colleges, in terms of number of hours the teachers are well prepared. In fact fewer than 2% reported no degree.

Mathematics Teachers. In this study only 7% of the mathematics teachers were without preparation in college mathematics. A study of individual teachers' schedules indicated that the mathematics course most frequently taught by teachers who had taken no college mathematics was one class in general mathematics; and in no case was he devoting full time to the subject.

The mathematics courses studied were divided in two categories: (1) the mathematics below, or prerequisite to, the calculus and (2) the calculus and more advanced courses. Although the mathematics teachers had an average of 23 semester hours in mathematics and 33% had masters' degrees, 39% had not had mathematics as advanced as the calculus. There was a large group of teachers well prepared in number of hours of mathematics and another large group with meager preparation. In many cases the teachers had not had recent courses in mathematics; one teacher in five had taken all their courses prior to 1950. The average number of hours in education was 31 compared with 23 in mathematics.

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* Kenneth E. Brown and Ellsworth S. Obourn, Qualifications and Teaching Loads of Mathematics and Science Teachers, Superintendent of Documents, Government Printing Office, Washington 25, D.C.

Science Teachers. Of the science teachers, 64% had the baccalaureate degree with an average of 47 semester hours in science. The average number of hours in one state was 56 and in another state 39. The semester hours of science credit were distributed as follows: chemistry, 17; biology, 14; physics, 9.5. The science teachers had taken 36 semester hours in education.

Teachers' Marks. Approximately 70% of both the science teachers and mathematics teachers had marks of "B" or above. About one-third had marks of "A−" or "B+." This study indicates that teachers' marks in general are quite respectable.

THE UICSM* TEACHER TRAINING FILMS†

GERTRUDE HENDRIX, University of Illinois, AND BYRL SIMS, NETRC§

The UICSM mathematics teacher-training films are designed to replace twenty sessions of a live demonstration class as a part of an intensive four-week teacher-training conference. The series consists of twenty half-hour films, prefaced by a shorter film which will introduce the class and show how the films were made. The films are *not* designed to teach the subject matter of the course to the teachers. They will study the course content in other conference sessions each day. The films, like live demonstrations, are intended for two major purposes:

1) An existence proof that the materials are teachable to junior high school children.
2) Vivid and repeated illustrations of a teaching method and teaching devices known to be successful with the UICSM FIRST COURSE materials.

Observers' familiarity with the content being taught is presupposed.

Eighteen of the films each trace the development of one topic through the entire course, showing important incidents of teacher and class caught on documentary film. Intervening time is covered by brief inserts of explanatory narrative. Such films were made possible by a specially-designed studio built into a regular classroom, with booths for three hidden cameras. One camera covers the teacher at all times; another provides a front view of the class; and the third takes a profile view. Each camera can "move in" for close-ups at appropriate moments. Each sequence used in the completed films is an edited combination from three rolls shot simultaneously.

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* University of Illinois Committee on School Mathematics.

† These films are made possible by a grant from the U. S. Office of Education to the National Educational Television and Radio Center, in cooperation with the University of Illinois. They were tested by use in a National-Science-Foundation-sponsored training conference during the past summer, 1960. The original grant finances the film making and the research study only. Provisions for distribution have not yet been made.

§ National Educational Television and Radio Center.

Such films are possible when (1) there is an outstanding teacher thoroughly familiar with good course content and with the kinds of situations likely to arise when one is teaching it; (2) there is a programmer also familiar with the course and aware of the topics with which a teacher new to the material is most likely to need help; (3) there is a class strong enough in prerequisite background not to require costly spur-of-the-moment digressions for remedial teaching of arithmetic; (4) there is another class in the same course taught by the same teacher and kept about two weeks ahead of the film class; and (5) there is a technical staff sensitive enough to what is needed to get camera coverage of the most significant spots, and to edit the film so that the things most valuable for teachers to see are retained.

Accounts of the programming process evolved for these films, and designing of the technical equipment to make them possible, are stories too long for a five-hundred word article.

THE MATHEMATICS PROGRAM AT THE UNIVERSITY OF PARANÁ

JAYME MACHADO CARDOSO, University of Paraná, Brazil

In the University of Paraná, as in all the Brazilian universities, the teaching of mathematics is done by each faculty (Sciences, Engineering, Economics) without following the American pattern of centralizing the teaching in one Department of Mathematics. Each faculty has its separate Department of Mathematics directly subordinated to the faculty's administration, practically without any connection among the different faculties.

The title of Bachelor in Mathematics is obtained in the Faculty of Sciences, where the student must be approved in the following fourteen subjects:

First year: 1) Mathematical analysis I, 2) Analytical and projective geometry; 3) Experimental physics I, 4) Vector calculus, 5) Philosophy.

Second year: 6) Mathematical analysis II, 7) Descriptive geometry, 8) Experimental physics II, 9) Tensor calculus, 10) Rational mechanics.

Third year: 11) Higher mathematical analysis, 12) Higher geometry (Topology), 13) Mathematical physics, 14) Celestial mechanics.

To remedy this inconvenience of the lack of connection and to give better possibilities of research, the Brazilian Government has given great incentive to the creation of Research Institutes in the universities. In 1959 there was founded in the University of Paraná, among others, the Institute of Mathematics. This institute is formed of four divisions: a) Division of Pure Mathematics, b) Division of Applied Mathematics, c) Division of Statistics, d) Division of Teaching and Divulgation. The institute is conducted by a director and by a council constituted of full professors in mathematics and statistics of all the faculties of the University.

Thanks to the existence of the Institute of Mathematics, it will be possible to give different courses which are not normally offered by the faculties of the University, especially in the modern part of mathematics.

THE METHODS COURSE IN MATHEMATICS FOR PROSPECTIVE SECONDARY SCHOOL TEACHERS

The editors of this department have invited a number of persons to describe the methods course for prospective secondary school teachers of mathematics that is offered by their institution. Recommendations of the Commission on Mathematics and other groups lay stress on the importance of the mathematics methods course. On the other hand, there is a great deal of discussion and considerable disagreement about what should be in a course. It is felt that these contributions will be helpful to those planning to establish such courses, or to improve those that are now in operation. Other contributions of this kind would be welcome and could be considered for later publication in this department.

Iowa State Teachers College. Among the several possible and defensible emphases the course might be given, we have chosen curriculum. This gives us the opportunities we seek to (a) review the objectives of instruction in mathematics; (b) look at typical organization of content, kindergarten through college; (c) question relative emphases on topics in light of changing emphases in mathematics and its applications; (d) identify contradictions, at least from the learner's point of view, in commonly taught language and concepts; (e) re-examine the roles of the teacher and the learner in light of new psychological insights into the learning process; (f) build hypothetical teaching sequences to illustrate and try out the principles evolved; (g) study critically the currently available experimental or recommended materials of instruction.

Of course one cannot set up these items, like hurdles in a track event, to be cleared one-by-one in the order *a* through *g*. A more gestalt approach is essential. Dewey said the man educated in a field is the one prepared to make sound judgment in that field. This is the sort of teacher we wish to incubate. The heat we use for this purpose is the active group, confronted with ideas and provided with situations that encourage critical thought.

H. C. Trimble

State University of Iowa. The undergraduate course in *Methods of Teaching High School Mathematics* as taught at the State University of Iowa consists, roughly, of two parts. The first part deals with the secondary curriculum—past, present, and future. The second part is devoted to special methods and techniques involved in teaching a variety of topics, *e.g.* development of operations with directed numbers. Emphasis is given to study and evaluation of current literature, particularly the journals and yearbooks of the National Council of Teachers of Mathematics and materials developed by the Commission on Mathematics, the School Mathematics Study Group, and committees at the Universities of Illinois and Maryland.

The trend in assigned reading is toward more content and less method. For majors this presents little or no problem; minors, on the other hand, often find it difficult to understand the new vocabulary with which they are confronted. As a result, it has become necessary to include in the course a varying amount of mathematical content. We believe, however, that a methods course should emphasize *method* rather than *subject matter.*

H. Vernon Price

University of California, Los Angeles. Fundamental Concepts of Mathematics—Algebra. This course was designed to give secondary teacher candidates the necessary maturity in algebra to make good high school teachers of algebra. Although it is not primarily a methods course, much attention is given to developing the topics discussed in ways which are suitable for high school presentation.

A thorough development of the number system is given, including a brief historical

account of some of the early number systems. A considerable amount of time is given to the consideration of number bases other than ten, including the changing of a numeral written in one base to the corresponding numeral written in another base. This is followed by the consideration of the eleven field postulates of algebra followed in turn by the postulational development of integers, positive and negative, rational numbers, and complex numbers. A brief treatment of irrational numbers completes the study of the number development of the number system.

Some consideration is given to mathematical proof including direct and indirect proof, and mathematical induction. A brief discussion is included on function and its representations.

A rather complete treatment of determinants and their applications is included, followed by an introduction to matrices, including the definition of rank and the multiplication theorem for matrices. The general theory of the solution of simultaneous linear equations, both homogeneous and non-homogeneous, is developed.

The last unit in the course treats the theory of algebraic equations in which are developed those theorems needed to determine the rational roots of a polynomial equation, together with rational approximations of the irrational roots.

Clifford Bell

University of Minnesota. Mathematics with its abstractness, its logical structure, its wide application, has unique learning problems. At one extreme it involves simple skills where memory and practice are essential. At the other extreme, the solution of a problem, the proof of a theorem, the application of a generalization requires a high level of creative thinking. Thus, the teacher of mathematics will need to know when and what concepts to teach, when and why students are having difficulty, when and how to make concepts meaningful, when and how to practice skills and to stimulate productive thinking. The better the teacher's background in mathematics, the greater his problem in selecting what he can hope to present to adolescents—hence, the need of a methods course.

The methods course in mathematics at the University of Minnesota includes topics such as the following: Motivating the learning of mathematics; guiding the student to discover new concepts; teaching the student how to study mathematics; building skill in solving problems; providing a program of acceleration and enrichment for gifted students; planning an effective review, remedial, and practice program; evaluating the learning of facts, skills, and problem solving; developing desirable attitudes and appreciations of mathematics; evaluating new curriculum recommendations; procuring and using new instructional materials; planning and demonstrating a variety of ways to present a mathematical topic; learning the goals of mathematics instruction; reading research in mathematics education.

The explosion of knowledge in the field of education, the expansion in the resources and materials for mathematics teaching, and the many new mathematics curriculum proposals make a methods course more important than ever. If as the Commission on Mathematics says, "A poor curriculum well taught is better than a good curriculum badly taught," then a methods course which improves teaching techniques is an essential course for every prospective mathematics teacher.

Donovan A. Johnson

University of Virginia. The methods course at the University of Virginia for prospective teachers of secondary school mathematics is built around four units: (1) Number and Computation, (2) Measurement, (3) Relationships and (4) Proof. There is no assigned textbook. The required readings at present include sections from the following: 13th, 20th, 22nd, 23rd and 24th *Yearbooks* of the N.C.T.M., the C.E.E.B.

Report, certain methods textbooks, and certain selected articles. For individual projects, a list of supplementary references is included with each unit.

The purposes of the course are as follows: (1) to trace briefly the development of these major concepts in grades one through twelve, with particular emphasis on their development in the upper six grades, (2) to illustrate methods which lead to a developmental (discovery) approach to learning mathematics, (3) to reorient these college students to secondary mathematics, and (4) to introduce them to various teaching materials.

Activities include written assignments on the units, reports, discussion, and sample lessons given by the students themselves. The students review some high school textbooks; special emphasis is placed now on their becoming acquainted with new text materials, such as those of the S.M.S.G., the U.I.C.S.M., and the Maryland Program. Near the end of the course, attention is given to evaluation of pupil progress. All student teachers at the University of Virginia, in addition to their work in methods courses, receive six hours instruction and laboratory practice in the use of visual aids.

William C. Lowry

CONTINENTAL CLASSROOM

Continental Classroom, the network TV program for college credit, will continue to focus on improving science instruction when it goes on the air for the third consecutive year September 26. The 1960–61 course, to be televised over NBC-TV, will be *Contemporary Mathematics*. The first semester will be devoted to *Modern Algebra;* the second semester, to *Probability and Statistics*.

To encourage credit enrollment, a new format for Continental Classroom has been created. *Modern Algebra* will be divided into two sections. College and university students seeking undergraduate credit will be required to view the lessons telecast on Monday, Wednesday, and Friday from 6:30 to 7 A.M. Teachers and others enrolled for graduate credit in the School of Education will be required to view the telecasts five days a week. The additional TV sessions on Tuesday and Thursday will be devoted to the teaching of modern algebra in secondary schools. The same pattern will be followed during the second semester when *Probability and Statistics* is offered.

Co-sponsors of *Contemporary Mathematics* are the Conference Board of the Mathematical Sciences, Learning Resources Institute, and the National Broadcasting Company. An eight-man advisory committee, headed by Dr. E. G. Begle of Yale University has been appointed by the Conference Board.

National teacher of *Modern Algebra* (selected by the advisory committee) will be Dr. John L. Kelley, professor of mathematics and head of the mathematics department at the University of California, Berkeley. Dr. Frederick Mosteller of the statistics department at Harvard University will teach *Probability and Statistics*. For the teacher-education programs of each course, Dr. Julius H. Hlavaty, chairman of the mathematics department at the DeWitt Clinton High School, New York, and Dr. Paul C. Clifford, associate professor at Montclair State College, Upper Montclair, N. J., will assist the national teachers.

As outlined by Dr. Kelley, *Modern Algebra* will include the fundamental concepts underlying the recent changes and developments in both the approach to and teaching of this subject. It will supply all prerequisites for the second semester offering in *Probability and Statistics*. A textbook, *Modern Algebra*, specially written by Dr. Kelley for this TV course, is available in bookstores.

ETS Cooperative Plan for Admission and Guidance*

A new plan for communication between high schools and colleges, enabling schools

* From a news release by the Educational Testing Service, Princeton, N. J.

to convey to the college a student's total high school record and other essential information about him, has been developed by Educational Testing Service of Princeton, N. J., and adapted for their own needs by educators in the state of Georgia.

The new "Cooperative Plan for Guidance and Admission" was announced jointly by Henry Chauncey, President of Educational Testing Service; Claude Purcell, State Superintendent of Schools in Georgia; and Harmon W. Caldwell, Chancellor of The University System of Georgia.

Recent developments in electronic data processing at Educational Testing Service now make it possible to report a student's total school performance—grades, activities, interests, health record, personal characteristics—in a form that is meaningful to employers and college admissions officers. The same process also makes feasible a reporting back to high schools of how well their graduates are doing in college and on the job.

As the first state to adapt the plan for its own needs, Georgia is a pilot state in implementing the system described by Dr. Chauncey as "holding promise for nationwide improvement of guidance in the high school and of the transition from school to college." Educators in several other states are currently studying the plan and considering ways to put it to work in their schools and colleges.

Trends in Engineering Enrollments

A recent report from the U. S. Office of Education indicates that undergraduate enrollment in the nation's 234 engineering colleges and universities declined 5.4% in the fall of 1959 in the face of an all-time high in general college enrollment. General college enrollments for this academic year show an increase of over 4%. On the other hand, the number of Master's degrees awarded in engineering in the United States increased 16% in 1959 over the preceding academic year. This is substantially higher than most advance estimates. The total number of Master's degrees in engineering awarded in 1959 is 6,761, compared with 5,788 in the year previous.

Bachelor's degrees in engineering totaled 38,162 in the academic year 1958–59, an increase of 10% over the preceding year. Doctor's degrees totaled 713, which is also an increase of 10%.

CBA Chemistry Project

A new chemistry course is being tried out in nine of the nation's high schools this year. The course has been under development by a group of high school and college chemistry teachers working on it for the past year. The present trial has its headquarters at Earlham College, Richmond, Indiana, and is supported by a grant from The National Science Foundation.

A major emphasis of the new course is concern with the atomic and molecular structures of substances. In these structures, the atoms are held together by chemical bonds which arise from the electronic nature of atoms. For this reason, the project is called The Chemical Bond Approach. Considerable attention is given to the relation between the geometrical properties of the underlying structure of a substance and its observable properties. This is proving to be not only a powerful conceptual tool for theoretical chemistry, but also an effective teaching device for introducing students to chemistry.

This year's trial is providing a basis for a revision of classroom and laboratory materials. One hundred teachers will be trained in the new course this coming summer. An expanded trial will be carried on next year.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1426. *Proposed by C. F. Pinzka, University of Cincinnati*

Professor E. P. B. Umbugio is trying to supplement his meager academic salary by entering soap contests. One such contest requires the contestants to find the number of paths in the following array which spell out the word MATHEMATICIAN:

```
                                M
                              M A M
                            M A T A M
                          M A T H T A M
                        M A T H E H T A M
                      M A T H E M E H T A M
                    M A T H E M A M E H T A M
                  M A T H E M A T A M E H T A M
                M A T H E M A T I T A M E H T A M
              M A T H E M A T I C I T A M E H T A M
            M A T H E M A T I C I C I T A M E H T A M
          M A T H E M A T I C I A I C I T A M E H T A M
        M A T H E M A T I C I A N A I C I T A M E H T A M
```

Umbugio has counted 1587 paths which originate from one of the first five rows. With the deadline for submitting entries approaching, he is distraught to say the least. Help the Professor out by finding the number of paths with a minimum of computation.

E 1427. *Proposed by F. Leuenberger, Zuoz, Switzerland*

If a_1, a_2, a_3 are the sides and h_1, h_2, h_3 the altitudes of a triangle T, show that

$$\sqrt{3}\sum_{1}^{3} a_i \geqq 2\sum_{1}^{3} h_i,$$

with equality if and only if T is equilateral. (Dedicated to the memory of Victor Thébault.)

E 1428. *Proposed by Aaron Lieberman, The G. C. Dewey Corporation, N. Y.*

Evaluate

$$\prod_{0}^{\infty} [1 + (1/2)^{2^n}].$$

E 1429. *Proposed by J. S. Brock, David Taylor Model Basin, Washington, D. C.*

If

$$f(\theta) = \frac{1 + a^2b - 2ab\cos\theta + 2b\cos 2\theta - a(1+b)\cos 3\theta}{1 + a^2 - 2a\cos 3\theta},$$

where $-1 < a < 1$, show that $\int_0^{\pi} f(\theta)d\theta = \pi$.

E 1430. *Proposed by M. S. Klamkin, AVCO Research and Advanced Development*

What is the highest order of multiplicity a root can have for the equation

$$x(x-1)(x-2)\cdots(x-n+1) = \lambda?$$

SOLUTIONS

An Inequality for the Triangle

E 1272 [1957, 432; 1958, 123]. *Proposed by the late Victor Thébault, Tennie, Sarthe, France*

If A, B, C are the angles of a triangle, show that

$$(\sin A/2 + \sin B/2 + \sin C/2)^2 \leqq \cos^2 A/2 + \cos^2 B/2 + \cos^2 C/2.$$

II. *Solution by Leon Bankoff, Los Angeles, Calif.* If the vertices are suitably named, all plane triangles belong to one or the other of the classes $A \geqq 60° \geqq B \geqq C$ or $C \geqq B \geqq 60° \geqq A$. In either event,

$$(\sin B/2 - 1/2)(\sin C/2 - 1/2) \geqq 0,$$

with the result that

$$4\sin B/2 \sin C/2 \geqq 2(\sin B/2 + \sin C/2) - 1.$$

Consequently,

$$1 + 4\Pi \sin A/2 \geqq 2(\sin A/2)(\sin B/2 + \sin C/2) + (1 - \sin A/2).$$

Now, if angle A remains constant, $\sin B/2 \sin C/2$ reaches a maximum when $B/2 = C/2 = (\pi - A)/4$. Hence

$$\sin B/2 \sin C/2 \leqq \sin^2[(\pi - A)/4] \equiv (1 - \sin A/2)/2.$$

It follows that

$$\Sigma \cos A \equiv 1 + 4\Pi \sin A/2 \geqq 2\Sigma \sin A/2 \sin B/2$$

or $\Sigma \cos^2 A/2 \geqq (\Sigma \sin A/2)^2$.

Editorial Note. Bankoff pointed out that the earlier solution [1958, 123] to this problem is faulty. This earlier solution states that "the minimum value for $\cos^2 A/2+\cos^2 B/2+\cos^2 C/2$ for $A/2+B/2+C/2 \leqq \pi/2$ is attained when $A/2=B/2=C/2$." The counter-example $A=10°$, $B=10°$, $C=160°$ shows this statement to be false; here $\Sigma \cos^2 A/2=2.0149$, whereas with $A=B=C$ we have $\Sigma \cos^2 A/2=2.25$.

Some Properties of the Fibonacci Numbers

E 1396 [1960, 81]. *Proposed by* (1) *Dustan Everman, Arizona State College,* (2) *A. E. Danese, Union College,* (2) *K. Venkannayah, Government College, Mercara, India*

For the Fibonacci sequence $\{f_n\}$, where $f_1=f_2=1$, $f_n=f_{n-1}+f_{n-2}$, $n \geqq 3$, show that

(1) Every fifth number in the sequence is divisible by 5.

(2) $f_{n+h}f_{n+k}-f_nf_{n+h+k}=(-1)^nf_hf_k$. (All the identities in Ganis, *Notes on the Fibonacci sequence*, this MONTHLY, vol. 66, p. 129, are special cases of this.)

(3) $$H(f_1, \cdots, f_{2n-1}) = \begin{vmatrix} f_1 & f_2 & \cdots f_n \\ f_2 & f_3 & \cdots f_{n+1} \\ \vdots & \vdots & \cdots \vdots \\ f_n & f_{n+1} & \cdots f_{2n-1} \end{vmatrix} = 0, \qquad n \geqq 3.$$

Solution by E. M. Scheuer, Space Technology Laboratories, Los Angeles, Calif.

(1) This follows since $f_5=5$ and $f_n=5f_{n-4}+3f_{n-5}$.

(2) Using the well-known formula $f_N=(a^N-b^N)/2^N\sqrt{5}$, where $a=1+\sqrt{5}$ and $b=1-\sqrt{5}$, and noting that $a^Nb^N/2^{2N}=(-1)^N$, one easily establishes the desired identity.

(3) Since the third row is the sum of the first two rows, this is obvious.

Also solved by A. N. Aheart, Alan Beal, J. H. Biggs, A. P. Boblétt, D. A. Breault, Brother Joseph Heisler, J. L. Brown, Jr., P. L. Chessin, F. B. Correia, D. F. Criley, A. E. Danese, T. N. Delmer, C. W. Dodge, Underwood Dudley, S. H. Eisman, Jane Evans, Dustan Everman, Evelyn Frank, Stanley Franklin and Gerald Hutchison (jointly), B. K. Gold, Michael Goldberg, L. D. Goldstone, H. W. Gould, S. H. Greene, Robert Heller, A. F. Horadam, Geoffrey Horrocks, A. R. Hyde, Shigeru Ishii and Joel Levy (jointly), Anne L. Johannsen, John Jordan, C. H. King, M. S. Klamkin, Donald Knuth, P. S. Landweber, Harry Langman, Barry MacKichan, D. C. B. Marsh, Marvin Mielke, E. P. Miles, Jr., John Moon and Ted Petrie (jointly), D. A. Moran, D. L. Muench, Katherine E. O'Brien, C. S. Ogilvy, Walter Penney, J. L. Pietenpol, C. F. Pinzka, Thomas Porsching, Joel Rascoff, B. E. Rhoades, R. M. Rippey, Barbara Sakitt, Benjamin Sapolsky, R. E. Shafer, R. T. Shannon, Vencil Skarda, C. V. L. Smith, Mrs. K. L. Smith, M. F. Stilwell, E. A. Sturley, Anna and R. G. Thompson (jointly), K. Venkannayah, Julius Vogel, Chih-yi Wang, T. West, C. T. Whyburn, Alan Wilson, Dale Woods, A. W. Yonda, and David Zeitlin.

Other proofs are possible. Thus (1) follows from (2) by taking $n=h=5$, $k=5a$, and then using induction. Identity (2) may be established by induction on n with h fixed followed by induction on h. Results (1) and (2) are not new (see Dickson, *History of the Theory of Numbers*, vol. 1, Chap. XVII).

Another Inequality for the Triangle

E 1397 [1960, 82]. *Proposed by Leon Bankoff, Los Angeles, California*

Show that $\Sigma AI \leqq \Sigma AH$, where I is the incenter and H the orthocenter of a triangle ABC.

Solutions by the proposer. I. It has been proved that $(\Sigma \sin A/2)^2 \leqq \Sigma \cos^2 A/2$ (see above solution to Problem E 1272). Hence

$$2\Sigma \sin A/2 \sin B/2 \leqq \Sigma \cos^2 A/2 - \Sigma \sin^2 A/2 = \Sigma \cos A = 4(\Pi \sin A/2) + 1.$$

Then

$$2\Sigma \csc A/2 \leqq 4 + \Pi \csc A/2 = 4 + 4R/r,$$

or

$$\Sigma AI \leqq 2(R + r) \leqq \Sigma AH.$$

II. Let P denote the orthogonal projection of I upon BC, and I_a the excenter relative to the vertex A. D, the midpoint of II_a lies on the circumference. The vertices may be named so that $C \geqq B \geqq 60° \geqq A$ or $A \geqq 60° \geqq B \geqq C$, depending on the species of triangle under consideration. In either case, we have $(BI-2r)(CI-2r) \geqq 0$, or $2r+(BI)(CI)/2r \geqq BI+CI$. Now, in the similar right triangles BIP and II_aC, we have $(BI)(CI)/2r = ID$. Hence $2r+ID \geqq BI+CI$. But $2R-ID \geqq AI$. It follows that $\Sigma AI \leqq 2R+2r \equiv \Sigma AH$, where, if the triangle is obtuse, the segment joining H to the obtuse angle is taken in the negative sense.

Also solved by L. D. Goldstone and D. C. B. Marsh.

Editorial Note. The proposer's Solution I led him to examine critically the earlier proof of E 1272 [1958, 123]. The detection of an oversight in this proof then led him to construct Solution II (which is independent of the result of E 1272) and, finally, in order to validate his Solution I, to the construction of the correct proof of E 1272 given above in this issue. Just as E 1397 can be obtained from E 1272, so can E 1272 be obtained form E 1397.

The Spice of Life

E 1398 [1960, 82]. *Proposed by A. N. Aheart, West Virginia State College*

If A, B, C are the angles of a triangle, show that $\cos A+\cos B+\cos C<2$.

I. *Solution by W. O. J. Moser, University of Manitoba.* We have $A+B+C=\pi$ and two of the angles, say A and B, acute. Hence

$$\begin{aligned}\cos A + \cos B + \cos C &= 2\cos (A + B)/2 \cos (A - B)/2 + \cos C \\ &\leqq 2\cos (A + B)/2 + \cos C \\ &= 2\cos (\pi - C)/2 + \cos C \\ &= 2\sin C/2 + 1 - 2\sin^2 C/2 \\ &= 3/2 - 2(\sin C/2 - 1/2)^2 \leqq 3/2,\end{aligned}$$

with equality only if $\cos(A-B)/2=1$ and $\sin C/2=1/2$, that is, only when the triangle is equilateral.

II. *Solution by A. R. Hyde, Bronxville, N. Y.* Let $y=\cos A+\cos B+\cos C=\cos A+\cos B-\cos(A+B)$. For any specified B, y is a maximum when $A=(\pi-B)/2$. The locus of these maxima is $y=2\cos(\pi-B)/2+\cos B$, for which, in turn, the maximum occurs when $B=\pi/3$. Hence maximum y is $3\cos(\pi/3)=3/2$, which occurs only when the triangle is equilateral.

III. *Solution by G. E. Forsythe and G. Szegö, Stanford University.* Consider A, B, C as variables in the closed interval $[0, \pi]$ with side condition $A+B+C=\pi$. Let A, B, C be a system for which $\max(\Sigma \cos A)$ is attained. If, say, $A=0$, then $B+C=\pi$ and $\Sigma \cos A=1$; if $A=\pi$, $B=C=0$ and again $\Sigma \cos A=1$. So we may assume that A, B, C are in the open interval $(0, \pi)$. Hence, using Lagrange multipliers, $-\sin A=-\sin B=-\sin C=\lambda$, so that $A=B=C$. (We cannot have $A=\pi-B$, since that would make $C=0$.) It follows that $\Sigma \cos A \leqq 3/2$, with equality only when the triangle is equilateral.

IV. *Solution by Wayne Shepherdson, Modesto Junior College.* One need consider only acute triangles. Set $\alpha=\pi-2A$, $\beta=\pi-2B$, $\gamma=\pi-2C$. In a semicircle of unit radius draw two radii such that three central angles α, β, γ are formed. Label the chords opposite these angles u, v, w, respectively. In the three isosceles triangles thus formed, each angle adjacent to u is A; adjacent to v is B; adjacent to w is C. Then $u=2\cos A$, $v=2\cos B$, $w=2\cos C$. Now $u+v+w$ is a maximum when u, v, w are three consecutive sides of a regular hexagon. It follows that $\Sigma \cos A \leqq 3/2$, with equality only when the triangle is equilateral.

V. *Solution by P. D. Thomas, Coast and Geodetic Survey, Washington, D. C.* On pages 186, 191 of R. A. Johnson, *Modern Geometry*, are found the relations

$$d^2 = R(R-2r) \quad \text{and} \quad \cos A+\cos B+\cos C = 1+r/R,$$

where r, R are the inradius and circumradius and d is the distance between the incenter and the circumcenter. From the first relation it follows that $2r \leqq R$, or $1+r/R \leqq 3/2$. Hence, from the second relation, $\Sigma \cos A \leqq 3/2$, with equality only when $d=0$, in which case the triangle is equilateral.

VI. *Solution by David Zeitlin, Remington Rand Univac.* One need consider only acute triangles. Applying the Erdös-Mordell inequality (this MONTHLY, [1958, 521]) to the center O of the circumscribed circle of radius R of the triangle ABC, we obtain $3R \geqq 2(r_1+r_2+r_3)$, where $r_1=R\cos A$, $r_2=R\cos B$, $r_3=R\cos C$. Thus $\Sigma \cos A \leqq 3/2$, with equality only for $A=B=C$.

VII. *Solution by D. C. B. Marsh, Colorado School of Mines.* Consider the sides of triangle ABC as vectors α, β, γ with vector sum zero and lengths a, b, c. Using the definition of "dot product" (*e.g.*, $\alpha \cdot \beta = -ab\cos C$) one finds

$$\cos A+\cos B+\cos C = (3-|\alpha/a+\beta/b+\gamma/c|^2)/2 \leqq 3/2.$$

Also solved by R. G. Albert, H. D. Arnett, D. W. Bailey, J. W. Baldwin, Leon Bankoff, Edward Barbeau, J. G. Báron, Alan Beal, Ralph Beals, H. F. Bechtell, E. E. Bosman, Marian Brashears, D. A. Breault, A. M. Broshi, C. K. Brown, J. L. Brown, Jr., W. J. Carpenter, F. B. Correia, R. W. Cottle, Thomas Curry, B. R. Davis, G. DiAntonio, Jane Evans, J. A. Faucher, M. P. Fobes, C. E. Franti, George Glauberman, Michael Goldberg, J. H. Goldman, L. D. Goldstone, R. H. Gramann, S. H. Greene, C. A. Grimm, Emil Grosswald, David Haley, Edward Harris, Geoffrey Horrocks, Shigeru Ishii and Joel Levy (jointly), R. A. Jacobson, John Jordan, Geoffrey Kandall, Irving Katz, J. M. Kingston, Eudon Kirkland, M. S. Klamkin, Donald Knuth, Joseph Konhauser, L. C. Kurtz, M. A. Laframboise, Harry Langman, J. F. Leetch, F. Leuenberger, R. J. Lewyckyj, H. C. Machin, Marvin Mielke, Q. G. Mohammad and M. R. Puri (jointly), C. S. Ogilvy, Walter Penney, W. M. Perel, J. L. Pietenpol, C. F. Pinzka, Bob Ramsey, Joel Rascoff, B. E. Rhoades, R. M. Rippey, Barbara Sakitt, Benjamin Sapolsky, R. E. Shafer, C. M. Sidlo, Vencil Skarda, L. L. Sleizer, Philip Smedley, E. L. Spitznagel, Jr., M. F. Stilwell, D. D. Strebe, Dmitri Thoro, J. A. Todd, J. H. Wahab, Chih-yi Wang, Chung-lie Wang, T. West, Charles Wexler, Bill Wise, Roscoe Woods, C. C. Yalavigi, and the proposer. Late solution by William Becker.

There were many ingenious and short solutions establishing the weaker inequality stated by the proposer. Several solvers located the problem (with the stronger inequality) as Prob. 25, 7th ed. of Hobson's *A Treatise on Plane and Advanced Trigonometry*. Many solvers offered more than one solution. Forsythe and Szegö, in alternative solutions, showed that $\Sigma \cos A = 3/2 - \Sigma(b-c)^2(b+c-a)/2abc$ and $\Sigma \cos A = 3/2 - d^2/2R^2$ (see Solution V for notation).

Counterfeit Coins

E 1399 [1960, 82]. *Proposed by H. S. Shapiro, New York University*

Counterfeit coins all weigh 9 grams and genuine coins all weigh 10 grams. One is given five coins of unknown composition, and an accurate scale (not a balance). How many weighings are needed to isolate the counterfeit coins?

Solution by N. J. Fine, Institute for Advanced Study. We shall show how to determine four coins in three weighings. In the following matrix, a 1 in the ith row and jth column denotes that the jth coin appears in the ith weighing:

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}.$$

Let x_j be the weight of the jth coin, less 9, so that $x_j = 0$ or 1. Thus we have the equations

$$\begin{aligned} x_2 + x_3 + x_4 &= s_1, \\ x_1 \quad + x_3 \quad\quad &= s_2, \\ x_1 + x_2 \quad\quad &= s_3. \end{aligned}$$

Adding, we determine x_4 from the congruence

$$x_4 \equiv s_1 + s_2 + s_3 \pmod 2.$$

We can then solve for x_1, x_2, x_3. To determine five coins in four weighings we need only weigh the fifth coin separately.

It can be proved that the determination of five coins in three weighings is impossible if the weighings are predetermined, that is, if we require a non-sequential solution. It is highly likely that it is impossible even without this restriction.

The ratio 3/4 of weighings to coins can be lowered to 5/7, using the matrix

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

It is an attractive conjecture that this ratio tends to 0 as the number of coins increases.

Also solved by C. W. Dodge, Michael Goldberg, M. S. Klamkin, Donald Knuth, D. C. B. Marsh, C. S. Ogilvy, Barbara Sakitt, and L. L. Sleizer.

Red and Green Balls

E 1400 [1960, 82]. *Proposed by S. D. Pratico, Iona College, New Rochelle, N. Y.*

There are 6 red balls and 8 green balls in a bag. Five balls are drawn at random from the bag and placed in a red box; the remaining 9 balls are put in a green box. What is the probability that the number of red balls in the green box plus the number of green balls in the red box is not a prime number?

Solution by C. W. Trigg, Los Angeles City College. If there are g green balls, $0 \leqq g \leqq 5$, in the red box, then there are $6-(5-g)$ or $g+1$ red balls in the green box, and $1 \leqq 2g+1 \leqq 11$. The only composite number in this span is 9, so the desired probability is $C(8, 4)C(6, 1)/C(14, 5)$ or 30/143, if unity is considered as prime. If unity is considered as not prime, the probability is $30/143+C(8, 0)C(6, 5)/C(14, 5)$ or 213/1001.

Also solved by A. N. Aheart, D. W. Bailey, J. W. Baldwin, Alan Beal, Ralph Beals, H. F. Bechtell, Roy Bertoldo, A. P. Boblétt, Julian Braun, A. W. Brunson, M. L. Cantor, R. W. Cottle, D. F. Criley, Monte Dernham, Underwood Dudley, Jane Evans, G. T. Fox, Charles Franti, B. K. Gold, Michael Goldberg, L. D. Goldstone, S. H. Greene, Emil Grosswald, W. C. Guenther, Geoffrey Horrocks, A. R. Hyde, John Jordan, Donald Knuth, Deena Ann Koniver, Morton Kupperman, W. E. Lawrence, R. J. Lewyekyj, L. I. Lowell, D. C. B. Marsh, Joseph Mayer, R. W. Means, Q. G. Mohammad and M. R. Puri (jointly), C. S. Ogilvy, J. T. Parent, Bart Park, Walter Penney, J. L. Pietenpol, C. F. Pinzka, Joel Raskoff, R. E. Shafer, Vencil Skarda, L. L. Sleizer, Philip Smedley, D. D. Strebe, Arthur Torell, Julius Vogel, J. K. Ward, Chung-lie Wang, W. R. Westphal, Charles Wexler, W. Zayachkowski, David Zeitlin, and the proposer.

Not all these solutions agreed.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers, The State University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers, The State University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

It is with real sorrow that we report the passing of a good friend and prolific contributor to the Problem Departments. His ingenious and challenging results in geometry and number theory have so identified him for many years with these columns that his name is the first to occur to many readers when the Department is spoken of. M. Victor Thébault died on March 19, 1960, at the age of 79, at his home in Tennie, Sarthe, France.

PROBLEMS FOR SOLUTION

4917. *Proposed by L. Lewin, Enfield, Middlesex, England*

Prove

$$\int_0^{\pi/6} \log^2(2\sin x)\,dx = 7\pi^3/216. \tag{1}$$

$$\int_0^{\pi/6} x\log^2(2\sin x)\,dx = 17\pi^4/25920. \tag{2}$$

4918. *Proposed by J. L. Ullman and C. J. Titus, University of Michigan*

Let α_k, β_k be complex numbers, $k=1, \cdots, n$, with $|\alpha_k| = |\beta_k| = 1$. Let $0 \leqq \arg \alpha_1 < \cdots < \arg \alpha_n < 2\pi$, $0 \leqq \arg \beta_1 < \cdots < \arg \beta_n < 2\pi$, where $0 \leqq \arg \gamma < 2\pi$ for any nonzero complex number γ. Also let $\sum_1^n \alpha_k = \sum_1^n \beta_k = 0$. Prove that $\sum_1^n \alpha_k \bar{\beta}_k \neq 0$.

4919. *Proposed by J. L. Ullman and C. J. Titus, University of Michigan*

Let $\phi(\theta)$ be real valued for $0 \leqq \theta \leqq 2\pi$, let $\phi'(\theta)$ be continuous and positive, let $\phi(0) = 0$, $\phi(2\pi) = 2\pi$ and let $\int_0^{2\pi} e^{i\phi(\theta)} d\theta = 0$. Prove that

$$\left| \int_0^{2\pi} e^{i(\phi(\theta)-\theta)} d\theta \right| > 4.$$

4920. *Proposed by Smbat Abian, University of Pennsylvania*

An Ab-integration in a ring A is a mapping $a \to a^*$ of A into itself satisfying the following conditions:

$$(a + b)^* = a^* + b^*, \qquad (a^*b + ab^*)^* = a^*b^* \qquad \text{for any pair } a, b \in A.$$

Is there a nontrivial Ab-integration in the field of real numbers?

4921. *Proposed by David Gale, Brown University*

Prove that the numbers $w_1, \cdots, w_8$ can be the numbers of games won by the eight teams at the end of a major league baseball season if and only if

$$\sum_{i=1}^{8} w_i = 616, \qquad \sum_{i=1}^{k} w_i \leqq 11k(15 - k), \qquad k = 1, \cdots, 7.$$

(During a major league season every two teams play each other 22 times.)

4922. *Proposed by Peter Crawley, California Institute of Technology*

In the solution of a recent problem (4761 [1959, 67]) it was shown that the additive group of rationals has no maximal subgroups. Prove the generalization: An Abelian group G has no maximal subgroups if and only if G is divisible (*i.e.*, $nG = G$ for all integers n).

SOLUTIONS

A Coin Tossing Probability

4864 [1959, 728]. *Proposed by D. J. Newman, Brown University*

Let a coin be tossed repeatedly, and let p be the probability that at some point the number of heads exceeds twice the number of tails. Prove

$$p = \tfrac{1}{2}(\sqrt{5} - 1). \tag{1}$$

I. *Solution by J. S. Frame, Michigan State University.* More generally, given the integers $a > b > 0$, let $f(t)$ be the probability that at some time in an infinite succession of coin tosses the number of heads exceeds a/b times the number of tails by more than t/b. If $z_1, \cdots, z_b$* denote the roots z such that

$$z^{-b} + z^a = 2 \qquad |z| < 1, \tag{2}$$

and if h_t denotes the sum of the $\binom{b+t}{t}$ distinct products of t factors with repetitions allowed, formed from the $b+1$ quantities $z_0 = 1, z_1 \cdots, z_b$, we shall prove that

$$p = f(0) = 1 - \prod_{i=1}^{b} (1 - z_i), \qquad f(t) = 1 - h_t(1 - p), \qquad t = 0, 1, 2, \cdots. \tag{3}$$

* The fact that the roots of (2) which are inside the unit circle are precisely b in number can be shown without difficulty by use of Rouchè's theorem.

Clearly for $a=2$, $b=1$, $t=0$, we shall then have

$$p^{-1}+p^2=2, \quad \text{or} \quad p^3-2p+1=(p-1)(p^2+p-1)=0, \tag{4}$$

and the given special case will have the solution $p=(\sqrt{5}-1)/2$.

Proof. The probability $f(t)$ is changed to $f(t-b)$ after a throw of heads, and to $f(t+a)$ after a throw of tails. Hence

$$f(t)=\tfrac{1}{2}[f(t-b)+f(t+a)]. \tag{5}$$

This difference equation of degree $a+b$ has $a+b$ linearly independent solutions of the form z_i^t corresponding to the $a+b$ roots of equation (2). However, since the probability $f(t)$ tends to 0 as t becomes positively infinite, only those solutions for which $|z_i|<1$ can be used. We write

$$f(t)=\sum_{i=1}^{b} c_i z_i^{b+t}, \qquad t \geqq -b; \qquad f(0)=p. \tag{6}$$

and determine the constants c_i by the boundary conditions

$$f(-1)=f(-2)=\cdots=f(-b)=1, \tag{7}$$

since for t negative the inequality is satisfied at the start. Elimination of the c_i from equations (6) and (7) gives the determinantal equation

$$\begin{vmatrix} f(t) & z_1^{b+t} & \cdots & z_b^{b+t} \\ 1 & z_1^{b-1} & \cdots & z_b^{b-1} \\ \cdot & \cdot & \cdots & \cdot \\ 1 & z_1 & \cdots & z_b \\ 1 & 1 & \cdots & 1 \end{vmatrix} = 0. \tag{8}$$

For $t=0$, $p=f(0)$, equation (8) expresses $1-p$ as the ratio of two Vandermonde determinants

$$1-p=V(1, z_1, z_2 \cdots z_b)/V(z_1, z_2, \cdots z_b)=\prod_{i=1}^{b}(1-z_i). \tag{9}$$

For $t=1, 2, \cdots$ equation (8) can be changed to

$$\frac{1-f(t)}{1-f(0)}=\begin{vmatrix} 1 & z_1^{b+t} & \cdots & z_b^{b+t} \\ 1 & z_1^{b-1} & \cdots & z_b^{b-1} \\ \cdot & \cdot & \cdots & \cdot \\ 1 & z_1 & & z_b \\ 1 & 1 & & 1 \end{vmatrix} \div \begin{vmatrix} 1 & z_1^{b} & \cdots & z_b^{b} \\ 1 & z_1^{b-1} & \cdots & z_b^{b-1} \\ \cdot & \cdot & \cdots & \cdot \\ 1 & z_1 & \cdots & z_b \\ 1 & 1 & \cdots & 1 \end{vmatrix} = h_t. \tag{10}$$

This function h_t is known from determinant theory to be the sum of the distinct products of t factors formed (with repetitions allowed) from $1, z_1, \cdots z_b$. This proves equation (3) and solves the problem.

It is interesting to note [see this MONTHLY, vol. 64, 1957, p. 238] that

powers of the b roots z_i which we have used are given by the convergent series

$$(11) \qquad z^q = w^q\left(1 + \sum_{k=1}^{\infty} \frac{q}{q+ks}\binom{(q+ks)/b}{k} w^{ks}\right), \qquad s = a + b, \qquad w^b = 1/2,$$

where the different values of w give the b different roots z.

II. *Solution by W. A. O'N. Waugh, McGill University.* Consider a particle which performs a random walk in accordance with the results of the coin-tossing game, by moving two places to the right for a tail and one place to the left for a head. If it starts from position $+1$, it reaches or goes to the left of the origin when the number of heads exceeds twice the number of tails. In this form the problem is a special case of an exercise in W. Feller, *An Introduction to Probability Theory and Its Applications* (2nd ed.) 1957, Ch. XIV, q. 3.

For a solution, let q_x be the probability of ultimate absorption at the origin if the particle starts from a position $x>0$. Then, with the convention that $q_0=1$, we have

$$q_x = \tfrac{1}{2}q_{x+2} + \tfrac{1}{2}q_{x-1} \qquad x = 1, 2, \cdots.$$

A trial solution $q_x=\lambda^x$ leads to $\lambda=1$ or $\frac{1}{2}(\pm\sqrt{5}-1)$. Obvious conditions on the solution are $q_0=1$, and $q_x\to 0$ as $x\to\infty$, and these give $q_x=2^{-x}(\sqrt{5}-1)^x$, whence the solution to the given problem is $q_1=\frac{1}{2}(\sqrt{5}-1)$.

Also solved by George Glauberman, Roger Pinkham, D. C. Stevens, and one whose name is missing.

The Inverse Tangent Integral

4865 [1959, 728]. *Proposed by L. Lewin, Enfield, England*

Defining the inverse tangent integral of the second order by

$$Ti_2(x) = \int_0^x \frac{\tan^{-1}(x)}{x}\,dx = \frac{x}{1^2} - \frac{x^3}{3^2} + \frac{x^5}{5^2} \cdots,$$

prove that $6Ti_2(1) - 4Ti_2(1/2) - 2Ti_2(1/3) - Ti_2(3/4) = \pi \log 2$.

Solution by M. S. Klamkin, AVCO Research and Development, Wilmington, Mass. The proposer has shown that

$$Ti_2\left(\frac{y^2}{2}\right) + \frac{1}{2}Ti_2\left\{\frac{y(2+y)}{2(1+y)}\right\} + \frac{1}{2}Ti_2\left\{\frac{2(y-1)}{y(2-y)}\right\} + Ti_2\left\{\frac{y}{2+y}\right\}$$
$$- Ti_2\left(\frac{y}{2-y}\right) + Ti_2\left(\frac{1}{1+y}\right) - Ti_2(y-1) = 2Ti_2(1) + \frac{\pi}{4}\log\left\{\frac{y(2-y)}{2(1+y)}\right\}.$$

(See L. Lewin, *Dilogarithms and Associated Functions*, London, 1958, p. 37). The desired result follows upon letting $y=1$.

Editorial Note. The problem and the proposer's solution were received before the publication of the book. A variety of similar formulas may be obtained, *e.g.*

$$Ti_2(7/24) + 2Ti_2(1/7) + 6Ti_2(1/3) - 8Ti_2(\tfrac{1}{4}) + Ti_2(3/4) + (\pi/2)\log(3/2) = 0.$$

Semi-magic Matrices

4866 [1959, 729]. *Proposed by F. D. Parker, University of Alaska*

Show that a semi-magic matrix A (the sums of the rows and columns are all equal) can be decomposed into a sum $B+C$ such that for integral K,

$$(B + C)^K = B^K + C^K.$$

Solution by Leonard Carlitz, Duke University. Setting aside the trivial solution $B=A$, $C=0$, let $A=(a_{rs})$ with $\sum_r a_{rs}=\sum_s a_{rs}=\lambda$, $r, s=1, 2, \cdots, n$; and let L be the n-rowed square matrix with every element unity. We get $AL=LA=\lambda L$, $L^2=nL$. Now let

$$B = \frac{\lambda}{n} L, \qquad C = A - \frac{\lambda}{n} L.$$

Then $BC=(\lambda/n)LA-(\lambda^2/n^2)L^2=(\lambda/n)\lambda L-(\lambda^2/n^2)nL=0$. Similarly $CB=0$. Hence by an easy induction

$$(*) \qquad A^k = B^k + C^k \qquad (k \geqq 1).$$

Note 1. The condition $k \geqq 1$ is necessary. For if (*) holds with $k=-1$, we get $(B+C)(B^{-1}+C^{-1})=I=(B^{-1}+C^{-1})(B+C)$, so that $I+BC^{-1}+CB^{-1}=0$, and $I+C^{-1}B+B^{-1}C=0$, and so $2I+(BC^{-1}+C^{-1}B)+(CB^{-1}+B^{-1}C)=0$.

On the other hand, (*) with $k=2$ yields $BC+CB=0$, so that $B^{-1}C+CB^{-1}=0$, $C^{-1}B+BC^{-1}=0$ and therefore $2I=0$.

Note 2. The above construction yields the trivial solution if and only if $\lambda=0$ or A is a constant multiple of L. It is not certain that in these cases a nontrivial solution exists.

Also solved by Gerald Goertzel, R. D. Gordon, and the proposer.

An Integral Inequality

4867 [1959, 729]. *Proposed by Ky Fan, Oak Ridge National Laboratory*

Let f be a measurable function such that $0<f(x)\leqq 1/2$ on $[0, 1]$. Prove

$$\frac{\exp\int_0^1 \log f(x)dx}{\int_0^1 f(x)dx} \leqq \frac{\exp\int_0^1 \log\{1-f(x)\}dx}{\int_0^1 \{1-f(x)\}dx}.$$

Solution by Tadeusz Stanisz, Jagiello University, Krakow, Poland. The inequality to be proved may be written in the form

$$(*) \qquad \log\left[\frac{\int_0^1 \{1-f(x)\}dx}{\int_0^1 f(x)dx}\right] \leqq \int_0^1 \log\left[\frac{1-f(x)}{f(x)}\right]dx.$$

Since the function $\phi(u)=\log\{(1-u)/u\}$ is convex for $0<u\leqq\frac{1}{2}$ (because $\phi''(u)\geqq 0$ for $0<u\leqq\frac{1}{2}$) the inequality (*) is an immediate consequence of a known form of Jensen's inequality. See, *e.g.*, Pólya u. Szegö, *Aufgaben und Lehrsätze*, II, 77.

Also solved by George Glauberman, and the proposer.

Editorial Note. The proposer's solution makes use of backward induction (see Hardy-Littlewood-Pólya, *Inequalities*, p. 20) and illustrates one of the rare situations where this method is naturally applicable.

A Special Infinite Series

4868 [1959, 729]. *Proposed by Ronald Pyke, Columbia University*

Show that for all $|\beta|\leqq e^{-1}$ and for all real x the function f defined by $f(0,\beta)=1$ and

$$f(x,\beta)=x\sum_{j=0}^{\infty}\beta^j\frac{(j+x)^{j-1}}{j!}\qquad(x\neq 0)$$

is finite and satisfies the relationship

$$f(x,\beta)=e^{xf(1,\beta)}.$$

Solution by Robert Weinstock, Oberlin College. We define

$$F(x,\beta)=\sum_{j=0}^{\infty}\beta^j\frac{(j+x)^j}{j!}\qquad(|\beta|<e^{-1}),$$

from which we obtain directly

$$f(x,\beta)=F(x,\beta)-\beta F(x+1,\beta).\tag{1}$$

But (*e.g.*, Pólya-Szegö: *Aufgaben und Lehrsätze*, III, 214)

$$F(u,\beta)=\frac{e^{uz}}{1-z}\qquad(|\beta|<e^{-1}),$$

where z is a certain solution of the equation $z=\beta e^z$. Thus, from (1),

$$f(x,\beta)=\frac{e^{xz}}{1-z}(1-\beta e^z)=e^{xz}.$$

Since, therefore $\beta f(1,\beta)=\beta e^z=z$, the desired result follows.

Also solved by S. H. Greene, M. S. Klamkin, Y. Matsuoka, and the proposer.

Editorial Note. The result has been noted before. Solvers supplied the following references: Bromwich, *Infinite Series*, p. 160, ex. 4; problems 4776 [1958, 783], 4489 [1953, 485].

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

Theory of Relativity. By W. Pauli, Pergamon Press, New York, 1959. xiv+241 pp. $6.00.

This is an English translation (G. Field) of the article "Relativitäts Theorie" in *Encyklopädie der Mathematischen Wissenschaften*, vol. V19, which first appeared in 1921. The original monograph has been supplemented by 25 pages of notes indicating later developments. Physical applications are kept to the fore, and the mathematical treatment is mature. Not suitable for a text (except in graduate seminars), this is an indispensable document for the physics bookshelf.

ARTHUR BERNHART
The University of Oklahoma

Advanced Complex Calculus. By Kenneth S. Miller. Harper, New York, 1960. 235 pp. $5.75.

This text is for a first course in the theory of functions of a complex variable for mathematics majors or engineers, and is written as a sequel to the author's *Advanced Real Calculus* (reviewed in this MONTHLY, vol. 64, 1957, p. 519). Titles of the chapters are: (1) Numbers and Convergence, (2) Topological Preliminaries, (3) Functions of a Complex Variable, (4) Contour Integrals, (5) Sequences and Series, (6) Calculus of Residues, (7) Some Properties of Analytic Functions, (8) Conformal Mapping, (9) Method of Laplace Integrals.

The first eight chapters furnish a compact coverage of the usual subject matter; the main theorems are carefully stated and proved, and are well arranged and indexed. Well-chosen, but not very numerous, exercises occur at each chapter end; they are more frequently extensions than illustrations of the text. One helpful feature of the book is the discussion of real multi-valued functions preliminary to the complex case. A less promising innovation is the last chapter. It appears that space of 14 pages is inadequate to discuss the solution of linear differential equations by contour integrals.

In general the exposition seems aimed at brevity rather than the large view. Many interrelations are left to the instructor to bring out, and he will need additional illustrations and exercises. Either as a reference book or text this volume should be a useful addition to the literature in this field.

E. H. CUTLER
Lehigh University

Real Analysis. By Edward James McShane and Truman Botts. Van Nostrand, Princeton, N. J., 1959. ix+272 pp. $6.60.

This excellent book presents, from the modern point of view, a large body of the fundamental theorems of real function theory. It contains eight chapters and three appendices. Chapter 0 is introductory and Chapter I introduces the real number system as a complete ordered field and ends with a rather nice discussion of various forms of the maximality principle. Chapter II starts immediately with the notion of directed convergence which plays a fundamental and unifying role throughout the book. Topological spaces are introduced here and are carried over to the next chapter where continuity is nicely treated. The next three chapters are centered mainly in the general Euclidean space and treat, among other things, functions of bounded variation, absolute continuity, differentiation, and integration. Of the two chapters on integration, the first introduces the Lebesgue-Stieltjes integral (via the Daniell-Darboux method) and concludes with the Riemann-Stieltjes integral. The second of these chapters presents the basic decomposition and substitution theorems. Concerning these two chapters, the reviewer feels that some improvement could have been made in pedagogy and selection; however, this is a matter of taste. The last chapter is devoted to linear spaces—chiefly the L_p spaces, and is, in fact, a good geometrical introduction to linear space theory. The treatment of Hilbert space is quite nice, especially the last section which presents the simultaneous spectral resolution of a finite number of commuting hermetian operators. Three appendices treat inductive definitions, six equivalent formulations of the maximality principle, and Tychonoff's Theorem (the latter is *not* presented as an equivalent formulation of the maximality principle). On the whole, the book is comprehensive, well written, and would make a good text for a solid course in analysis.

PASQUALE PORCELLI
Louisiana State University

Logic In Elementary Mathematics. By Robert M. Exner and Myron F. Rosskopf. McGraw-Hill, New York, 1959. ix+274 pages. $6.75.

Fundamentally, this is a textbook in symbolic logic covering the usual topics; the statement calculus, the restricted predicate calculus and proof and demonstration. In particular, the book uses examples from mathematics and illustrates the applications of logic in mathematics. The result is a book of considerable value for advanced undergraduates and graduate students. In a number of respects, the development is, in my opinion, too difficult for anyone not at least partially familiar with the concepts. There is some confusion in the discussion of the difference between names and names of names (pp. 12–13).

LOUIS O. KATTSOFF
Boston College

An Introduction to Mathematics. By Donald W. Western and Vincent H. Haag. Holt, Rinehart and Winston, New York, 1959. xi+580 pp. $7.50.

Writing "new-approach" books for freshmen has apparently become as compelling as writing calculus texts. The motivation for this activity seems clear enough: one adopts a text and another gets written as an outgrowth of notes prepared to explain the one adopted. This seems a reasonable way for new texts to emerge, and the result in this instance is reasonable, though we can well expect that someone—by the same inductive process—will produce another after adopting this.

The chapter titles pretty well tell the story of what the authors think freshmen should sample from, and the order in which the sampling should occur: Logic, The Natural Numbers, The Complex Number System, Extensions of the Logic of Algebra, Numerical Calculation, Algebra of Sets, Analytic Geometry and Functions, The Probability Function, Circular Functions, Lines and Vectors, Conic Sections, Limits of Sequences and the Definite Integral, Limits of Functions and the Derivative, The Calculus, Curve Tracing, Matrices, Analytic Geometry of Three Dimensions.

The presentation is generally good and at times novel. This remark, along with the table of contents, is meant to indicate that this is a fairly well-written, all-things-to-all-people textbook. It clearly merits consideration by anyone teaching modern freshman courses.

There are the usual displays of typographical errors and obvious omissions found in first printings. The detection and correction of these can be assigned as a valid class exercise.

Additionally, however, one finds things which will cause the student to wonder whether the authors were serious in their preface remark to the effect that an understanding of logic is a necessary tool for mathematics. As examples (by no means a complete list) of what occurs, one concludes from an example on page 26 that "Pope John is President of the United States" is a *meaningless* statement. An unfortunate example on page 53 indicates that each multiple of 2 is a multiple of 6, and from an example following a discussion of what it means to factor a polynomial, one can deduce that $x+2$ is a polynomial of degree less than one (pages 96–97). From the definition of a function (page 182) and an explanatory example (page 183), it follows that some week of December, 1959, is blessed with 31 days. Page 233 contains a discussion of combinations and tells us that "Two *combinations* are *different* if all their elements are not the same," and page 271 presents a delightful example of how logical reasoning aids mathematical study: "Many geometric and physical problems involve triangles, and every triangle can be considered as a pair of right triangles obtained by erecting a suitable perpendicular from a vertex to the opposite side of the triangle. Hence, we shall focus attention on right triangles."

Surely there is a great deal to say for the increasing practice of presenting mathematics with some unity and some attention to logical detail. In doing

this, however, is it really necessary to exclude or hide some important notions? (The idea of a prime and the unique factorization theorem are left as a library exercise.) And is it necessary that easy things become complicated? (The text uses induction to prove that $n(n+1)$ is even.)

It will, then, be necessary for the teacher to prepare complementary notes for use with this as a text. We can expect that this will result in a really good, corrected second edition as well as other texts by other authors.

ROBERT J. WISNER
Haverford College and
Institute for Advanced Study

Introduction to the Laplace Transform. By Dio L. Holl, Clair G. Maple and Bernard Vinograde. Appleton-Century-Crofts, New York, 1959. viii+174 pp. $4.25.

This material has been prepared as a text for a three-hour course on the quarter system. It is designed to follow a first course in differential equations and is more complete than the one-chapter or appendix coverage usually appearing in an engineering mathematics book. The theory is kept fairly elementary by restricting the transform variable to the reals. The theorems and proofs are carefully worded. Some harder theorems borrow results from the advanced calculus. References are given where proofs are omitted. The usual material is well covered. There is a good formal treatment of impulse functions and superposition theorems for the responses to a unit function input or to a unit impulsive function input. The applications are interesting and clearly discussed. References are given for the derivations of the differential equations involved. The second chapter of applications uses partial differential equations. The final chapter extends some theorems so that they hold for some functions having infinite discontinuities.

The problems appear to be well selected. Seven pages of transforms appear in the appendixes. It looks like a good text.

EARL LAFON
University of Oklahoma

Introduction to Probability and Statistics. By B. W. Lindgren and G. W. McElrath. Macmillan, New York, 1959. xiii+277 pp. $6.25.

Introduction to Probability and Statistics emphasizes ideas in statistics and statistical methods and their understanding. It is neither a cook-book of statistical recipes nor a text on mathematical statistics. The book provides an interesting, stimulating, and refreshing approach to the first course in statistics. Probabilities are considered as mathematical models, concepts of samples and populations are carefully discussed, elementary presentations of theories of testing hypotheses and estimation are accurate and readable.

Topics in this book are well organized and tied together without frequent

annoying back references. It is evident that the authors have given much thought to the presentation of most of the fundamental ideas and methods of statistics to the beginning student. Arguments are often intuitive and heuristic but convincing without being verbose. Emphasis in discussing statistical methods is placed on assumptions necessary for their valid applications. The usual topics are covered and a large number of procedures are included that are not usual in the introductory book. Subject matter includes basic probability, distributions, sampling, estimation and testing, tests on location and dispersion, introductory analysis of variance, regression, nonparametric methods, quality control and acceptance sampling, sequential analysis, and decision theory. Examples and problems are well chosen and graduated and relate to realistic applications. Beginning chapters have review problems at the end.

Introduction to Probability and Statistics should be considered seriously for the first course in statistics for the undergraduate. It is particularly suited for students of the physical sciences. Beginning calculus would assist the student but should not be a prerequisite. Most of the material could be covered in a one-semester course. A course based on this book should interest more students in the advanced study of statistics and this is needed.

RALPH A. BRADLEY,
The Florida State University

Introduction to Mathematical Statistics. By Robert V. Hogg and Allen T. Craig. Macmillan, New York, 1959. ix+245 pp. $6.75.

This new text is intended for a two-semester first course in mathematical statistics.

The authors state that their experience with a preliminary version indicates that it is desirable for the student to have some work beyond a first course in calculus, but that the content of this additional mathematical training is unimportant. Since elementary calculus will not suffice to derive much statistical theory, more advanced results are stated and used as appropriate. These results are given without proof save for about six pages devoted to set theory and set functions. The latter notions are exploited throughout the book.

Most of the "standard" topics of a first course are covered and there is no need to reproduce the table of contents but the following selection of items may indicate the "flavor" of the book. There is a good chapter on finding the distribution of a function of one or more random variables, including situations where the function is not one-to-one. The material on estimation is very good for a book at this level. Among the topics in this area one finds a discussion of sufficient statistics, complete classes of probability density functions, uniqueness, and the Koopman-Pitman class of probability density functions. Limiting distributions and limiting moment-generating functions are broached. Distributions of order statistics and functions thereof are discussed, the probability integral transform theorem is proved, confidence intervals for quantiles using order statistics are evaluated, and tolerance limits are introduced.

A good set of problems follows each section. Rather laconic tables of the normal, chi-square, "t," and "F" distributions are appended.

This is a carefully written text. The reviewer feels that if will prove successful in the classroom and that it can also be used profitably for independent study.

ERNEST M. SCHEUER
Space Technology Laboratories, Inc.

BRIEF MENTION

Homology Theory on Algebraic Varieties. By Dr. A. H. Wallace. Pergamon Press, New York, 1958. viii+115 pp. $5.50.

This book will be reviewed in the Spring 1960 *Bulletin of the American Mathematical Society* by Professor Stewart S. Cairns.

University Mathematics, (2nd ed.). By Joseph Blakey. Philosophical Library, New York, 1959. 581 pp. $10.00.

Not a book which will meet the current accepted American standards, either of rigor or teachability, but still a possible source for the tricky problems which apparently are still common on English degree-examination papers.

A Treatise on Algebraic Plane Curves. By Julian Lowell Coolidge. Dover, New York, 1959. xxiv+510 pp. $2.45.

A reprint of Coolidge's work written for Oxford Press, apparently about 1929.

The Theory of Numbers and Diophantine Analysis. By Robert D. Carmichael. Dover, New York, 1959. 118 pp. $1.35.

Carmichael's two books (1914 and 1915) bound as one volume.

Algebraic Theories. By Leonard E. Dickson. Dover, New York, 1959. ix+276 pp. $1.50.

This reprint of Dickson's *Modern Algebraic Theories*, first published in 1926, still contains much of interest. Our thanks to the publisher for very appropriately dropping the first word of its former title. The main concepts discussed center around matrices, invariance, and groups.

An Elementary Treatise on Fourier's Series. By William Elwood Byerly. Dover, New York, 1959. ix+287 pp. $1.75.

Byerly's 1893 treatise still may have historical interest for students of the history of mathematics. These students should also be reminded of the very excellent account of Fourier's series available as the first Slaught Paper, *Fourier's series: A genesis and evolution of a theory*, by R. E. Langer.

Handbook of Automation, Computation and Control, Vol. II. Edited by Eugene M. Grabbe, Simon Ramo and Dean E. Wooldridge. Wiley, New York, 1959. xxiii+1070 pp. $17.50.

This extensive volume is devoted to computers and data processing. Both digital and analog computers, their use and construction, are discussed by a staff of specialists.

Journal of the Association for Computing Machinery. The Association for Computing Machinery, 2 E. 63rd St., New York 21, N. Y. Quarterly, $6 per year, including ACM membership.

An increasing amount of applied mathematics is finding its way into this journal, and hence we call its attention to our readers.

Numerical Methods for Nuclear Reactor Calculations. By G. I. Marchuk. Consultants Bureau, Inc., New York, 1959. 295 pp. $60.00.

Although the price of this paperback, lithoprinted volume will put it beyond the price range for most, still it is interesting to note how much mathematics including Jacobians and partial differential equations is included in this Russian translation.

Quantum Chemistry Integrals and Tables. By J. Miller, J. M. Gerhauser and F. A. Matsen. University of Texas Press, Austin, 1959. 1221 pp. $15.00.

Designed to be used in Quantum Mechanical calculations involving modern internally programmed computers. These tables, computed and printed directly from IBM 650 records, should prove a valuable adjunct to any library.

A Table of the Incomplete Elliptic Integral of the Third Kind. By R. G. Selfridge and J. E. Maxfield. Dover, New York, 1959. xiv+805 pp. $7.50.

A set of 6-place tables prepared by the Research Department U. S. Naval Ordnance Test Station on their IBM 704 computer, carefully verified both by echo-checking and by means of a check integral at the completion of each value of a. Libraries will be well advised to obtain this set of tables while it is still available.

Sampling Inspection Tables (2nd ed.). By H. F. Dodge and H. G. Romig. Wiley, New York, 1959. xi+224 pp. $8.00.

In addition to the sampling tables, a collection of "probability-of-acceptance" curves is included. The table portion appears identical with a similar work published in 1944 by these authors.

NEWS AND NOTICES

EDITED BY LLOYD J. MONTZINGO, JR., *University of Buffalo*

Readers are invited to contribute to the general interest of this department by sending news items to L. J. Montzingo, Jr., Mathematical Association of America, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Professor Eckford Cohen, University of Tennessee, represented the Association at the inauguration of Dr. A. D. Holt as President of the University of Tennessee on May 13 and 14, 1960.

Professor H. B. Curry of the Pennsylvania State University mathematics department was honored by being named Evan Pugh Research Professor, effective February 1, 1960. These professorships, named after the first president of the University, were established by the Board of Trustees to recognize outstanding research by members of the faculty, and to provide support for the continuation of this research.

Professor William Feller, Princeton University, and Professor Antoni Zygmund, University of Chicago, have been elected members of the National Academy of Sciences.

Professor Magnus Hestenes, University of California, Los Angeles, was awarded an honorary Doctor of Science Degree by Wartburg College on June 7, 1960.

Mr. Stephen Lichtenbaum, Harvard University, has been awarded the William Lowell Putnum Prize Scholarship for the twentieth competition.

Dr. R. F. Rinehart, Director of Special Research and Operations Research at Duke University, was awarded the honorary degree of Doctor of Science by Wittenberg University on June 6, 1960.

Millsaps College: Professor T. L. Reynolds has been appointed Professor and Acting Head of the Department of Mathematics at William and Mary College; Professor S. R. Knox has been granted a sabbatical leave for the academic year 1960–61 to do advanced graduate work in Mathematical Statistics at Virginia Polytechnic Institute.

Western Michigan University: Assistant Professor Stanislaw Leja has been promoted to Associate Professor; Mr. J. D. Vollmer, Wayne State University, has been appointed Instructor.

University of Arizona: Drs. H. M. Lieberstein, Army Research Center, University of Wisconsin and Paul Slepian, Hughes Research Laboratories, Culver City, California, and Assistant Professor Berthold Schweizer, University of Southern California, have been appointed Associate Professors; Drs. J. D. Brooks, University of Southern California, and D. E. Myers, Millikin University, have been appointed Assistant Professors.

University of North Carolina: Associate Professor W. R. Mann has been promoted to Professor and will be on leave at the University of California, Berkeley, on a National Science Foundation Faculty Fellowship for the year 1960–61; Dr. C. W. Patty, University of Georgia, has been appointed Research Instructor; Assistant Professor R. L. Davis, University of Virginia, has been appointed Assistant Professor; Professor Rafael Artzy, Israel Institute of Technology, Haifa, Israel, has been appointed Associate Professor; Professor R. H. Sorgenfrey, University of California, Los Angeles, has been appointed Visiting Professor for the fall semester, 1960; Professor M. E. Shanks, Purdue University, has been appointed Visiting Professor for the spring semester, 1961; Assistant Professor J. H. Wells will be on leave at the University of California, Berkeley, for the year 1960–61.

University of South Carolina: Drs. E. E. Enochs, University of Chicago, and J. L. Boal, Massachusetts Institute of Technology, have been appointed Assistant Professors; Assistant Professor H. T. LaBorde, University of Cincinnati, has been appointed Associate Professor.

Mr. R. E. Blewster Jr., CONVAIR, Fort Worth, Texas, has accepted a position as Mathematician with the Air Force Flight Test Center, Edwards Air Force Base, California.

Mr. D. W. Bowan, Millikin University, has accepted a position as Assistant Research Mathematician at the University of Dayton Research Institute, Data Processing Division.

Mr. Juliun Braun, Seattle, Washington, has accepted a position as Electronic Programming Supervisor with the Chrysler Corporation, Missile Division, Detroit, Michigan.

Professor C. F. Brumfiel, Ball State Teacher's College, has been appointed Associate Professor at the University of Michigan.

Assistant Professor R. J. Buehler, Iowa State University, has been promoted to Associate Professor.

Mr. J. F. Burke, University of Maine, has been appointed Assistant Professor at Norwich University.

Dr. Harold Chatland, Montana State University, has accepted a position as Engi-

neering Specialist in the Electronic Defense Laboratory of Sylvania Electric Products, Mountain View, California.

Associate Professor J. G. Christiano, Northern Illinois University, has been promoted to Professor.

Mr. C. L. Conner, University of Colorado, has accepted a position as Systems Analyst with the Radio Corporation of America, Moorestown, New Jersey.

Mr. T. E. Depkovich has accepted a position as Technical Specialist—Dynamics with the Aerojet General Corporation, Sacramento, California.

Mr. D. C. Fried, General Electric Company, Phoenix, Arizona, has accepted a position as Scientist with Technical Operations, Incorporated, Washington, D. C.

Dr. Seymour Geisser has returned to his position as Mathematician with the National Institute of Health, Washington, D. C., after having served as Visiting Associate Professor at Iowa State University during the spring semester, 1960.

Assistant Professor John Hilzman, Harpur College, has been appointed Assistant Professor at Idaho State College.

Professor B. W. Jones, University of Colorado, has returned from his leave at the University of Puerto Rico.

Mr. Irving Kay, Burroughs Research Center, Paoli, Pennsylvania, has accepted a position as Project Leader with the International Electric Corporation, Paramus, New Jersey.

Mr. S. M. Keathley, Chance Vought Aircraft Company, Dallas, Texas, has accepted a position as Senior Numerical Analyst with Pratt & Whitney Aircraft, West Palm Beach, Florida.

Assistant Professor L. H. Lange, University of Notre Dame, has been appointed Assistant Professor at San Jose State College.

Associate Professor Anne L. Lewis, University of North Carolina, Woman's College, has been promoted to Professor and appointed Head of the Mathematics Department.

Mr. I. J. Lieberman, Philco Corporation, Philadelphia, Pennsylvania, has accepted a position as Manager of the Digital Systems Laboratory with the Kearfott Company, Clifton, New Jersey.

Dr. D. O. McKay, University of Buffalo, has been appointed Instructor at the University of Western Ontario, London, Ontario, Canada.

Associate Professor R. C. Meacham, University of Florida, has been appointed Professor at Florida Presbyterian College.

Mr. Kenneth Moosman, Dayton Air Force Depot, Dayton, Ohio, has accepted a position as Senior Mathematical Analyst with the Lockhead Missiles & Space Division, Sunnyvale, California.

Mr. R. F. Norris, Arizona State University, has accepted a position as Associate Physicist with the International Business Machines Research Laboratories, San Jose, California.

Associate Professor Ingram Olkin, Michigan State University, has been appointed Associate Professor at the University of Minnesota.

Associate Professor G. W. Patterson, University of Pennsylvania, has been appointed Vice-Chairman of the Electronic Computers Committee of the Institute of Radio Engineers.

Mr. Sylvester Reese, University of Wisconsin, has accepted a position as Research Engineer with the Autonetics Division of North American Aviation, Incorporated, Downey, California.

Associate Professor Charles Riggs, Texas Technological College, has been promoted to Professor.

Dr. A. S. Said, Columbia University, has accepted a position as Senior Chemist with Beckman Instruments, Incorporated, Fullerton, California.

Associate Professor Charles Saltzer, Case Institute of Technology, has been appointed Professor at the University of Cincinnati.

Dr. Harold Shulman, Institute of Mathematical Sciences, New York University, has accepted a position as Consultant with the Service Bureau Corporation, Brooklyn, New York.

Mr. C. D. Sise has accepted a position as Analyst in the Machine Computation Laboratory of the United Aircraft Corporation Research Laboratory, East Hartford, Connecticut.

Sister Mary Bonaventure, Immaculate Conception Junior College, has been appointed Teacher of Mathematics and Sciences at the St. Joseph High School, Camden, New Jersey.

Mr. M. M. Sluyter, Armour Research Foundation, Chicago, Illinois, has accepted a position as Research Scientist with the Advanced Reaserch Division of THERM, Incorporated, Ithaca, New York.

Mr. F. L. Thompson, University of California, Berkeley, has accepted a position as Mathematician at the Naval Ordnance Test Station, China Lake, California.

Mr. L. J. Watson, Jr., Phoenix Union High School, Phoenix, Arizona, has accepted a position as Head of Presentations with the System Development Corporation, Santa Monica, California.

Mr. S. H. Cunha, Quebec Hydro-Electric Commission, died March 21, 1960. He was a member of the Association for ten years.

Professor E. J. Finn, Catholic University of America, died May 9, 1960. He was a member of the Association for twenty-four years.

Professor A. L. Nelson, Wayne State University, died April 5, 1960. He was a charter member of the Association.

M. Victor Thébault, Tennie, Sarthe, France, died March 19, 1960. He was a member of the Association for twenty-eight years.

CONFERENCE ON MECHANICS

The Seventh Midwestern Conference on Fluid Mechanics and Solid Mechanics will be held at Michigan State University, East Lansing, September 6–8, 1961. Authors wishing to present papers should send in their abstracts preferably by January 1, 1961. For further information, address inquiries to: Professor J. E. Lay, Mechanical Engineering Department, Michigan State University, East Lansing, Michigan.

THE MATHEMATICAL ASSOCIATION OF AMERICIA

Official Reports and Communications

NEW MEMBERS

Professor Henry L. Alder, Secretary, announces that the following 292 persons have been elected to membership by the Board of Governors on applications duly certified.

CHARLES H. ACKERKNECHT, Student, Los Angeles State College

GORDON D. ADAMS, B.A. (Swarthmore) Asst., Cornell University

EDA S. ALTER, Student, Barnard College

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MRS. PATRICIA K. ANDRESEN, M.A. (Missouri) Instr., University of Missouri

LOUIS S. ARONICA, B.S. (Pennsylvania S.U.) Instr., Western College

DONALD G. ARONSON, Ph.D. (M.I.T.) Asst. Prof., University of Minnesota

DOUGLAS R. AUGUSTINE, B.A. (Washington) Engineering Technician, Diamond Ordnance Fuze Lab.

GEORGE L. BALDWIN, M. A. (Eastern New Mexico) Asst. Prof., New Mexico State University

MRS. RUTH BECK, B. A. (Stern Coll.) Grad. Asst., University of Pennsylvania

MICHAEL G. BECKER, Student, Xavier University

WILLIAM R. BECKER, Student, New York University; Programmer, Service Bureau Corporation

JOSEPH BEER, M. A. (Syracuse) Grad. Asst., Syracuse University

BENJAMIN R. BEH, Student, St. Ambrose College

JAMES C. BEIDLEMAN, M.S. (Bucknell) Research Asst., Ordnance Research Laboratory

PETER J. BICKEL, Student, University of California, Berkeley

THOMAS E. BLACK, JR., Teaching Asst., Purdue University

MRS. PATRICIA L. BLACKER, Student, San Jose State College

GEORGE R. BLAKLEY, M.A. (Maryland) NSF Fellow, University of Maryland

J. MITCHELL BLOSE, M.Ed (Pittsburgh) Chairman, Senior High School, Kittanning, Pennsylvania

WALTER BLUMBERG, M.A. (Brooklyn Coll.) Teacher, Flushing High School, Queens, New York

JAN L. BOAL, Ph.D. (M.I.T.) Asst. Prof., University of South Carolina

ELDON C. BOES, Student, St. Ambrose College

ROY B. BOGUE, B.E.E. (Auburn) Instr., Auburn University

ARVAL W. BOHN, M.A. (Texas) Instr., Louisiana State University in New Orleans

ROBERT E. BOHRER, Student, South Dakota School of Mines and Technology

VIROOL BOONYASOMBUT, B.S. (Chulalongkorn U., Thailand) Grad. Student, Ohio State University

R. MURRAY BOOTHROYD, M.Ed. (Minnesota) Instr., Sherburn High School, Minnesota

WARREN L. BOSCH, Student, Wittenberg University

MRS. DELORES H. BRIGHT, M.A. (Long Beach S.C.) Head of Dept., Narbonne High School, Long Beach, California

MARY A. BRIGHT, Student, Ohio State University

J. ROBERTS BRITTON, Supt., Hawaiian Electric Company; Lecturer, Dept. of Engineering, University of Hawaii

AVIAD M. BROSHI, Student, Yeshivah of Flatbush High School, Brooklyn, New York

ROLAND BROSSARD, M.Sc. (Montreal) Asso. Prof., University of Montreal

CALVIN K. BURGE, M.S. (Illinois) Instr., Lansing Community College

ENOCH BURTON, M.A. (Indiana) Head of Dept., Shortridge High School, Indianapolis, Indiana

AGOSTINO BURZIO, Student, University of Santa Clara

THOMAS V. BUSCEMI, B.S. (St. Francis Coll.) Teacher, Clayton High School, New Jersey

LORETO M. BUZZEO, Student, School of Education, Fordham University

ELWIN F. CAMMACK, M.Ed. (Brigham Young) Instr., Weber College

J. FRANCIS CAMPBELL, B.S. (Montana State Coll.) Grad. Asst., Montana State College

GERALD D. CHAPMAN, Student, University of California, Los Angeles

HARRY T. CHASE, JR., Student, University of Santa Clara

CHARLES E. CHRISTIAN, Student, Purdue University

CHARLES L. CHUSTZ, B.S. (Louisiana S.U.) Teacher, LaGrange Senior High School, Lake Charles, Louisiana

WILLIAM W. CLARK, B.A. (Reed Coll.) Teacher, Fremont Junior High, Monterey, California

MCLOUIS CLAYTON, Student, Shaw University

MOTHER M. ANCILLA CLEARY, M.S. (Fordham) Asst. Prof., College of New Rochelle

LT. FREDERICK H. CLEVELAND, B.S. (Texas A&M) U. S. Air Force

MAJOR CORNELIUS J. COMBER, U. S. Air Force

RICHARD S. COOKE, B.S. (Northeastern) Programmer, Sylvania Electric

WILLIAM SECORD COOPER, M.S. (M.I.T.) Grad. Student, University of California, Berkeley

WILLIAM E. COPPAGE, M.S. (Texas A&M) Asst. Instr., Ohio State University

HARRY S. COUZINS, B.A. (California, Berkeley) Chairman of Dept., Chaffey High School, Ontario, California

MRS. CAROLYN G. CRANDELL, Student, William Smith College

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ROBERT N. CROCKETT, Student, Church College of Hawaii

CHARLES C. DALTON, M.A. (Alabama) Physicist, Army Ballistic Missile Agency

EVMENIOS P. DAMON, M.S. (Drexel Inst. Tech.) Leader, Programming Group, Radio Corporation of America

HERBERT A. DAVID, Ph.D. (London) Prof., Statistics, Virginia Polytechnical Institute

THOMAS N. DELMER, Student, University of Santa Clara

THEODOSIUS L. DEMEN, Ph.D. (St. Louis) Asst. Prof., University of Dallas

BERNARD M. DONAHOE, M.A. (San Jose S.C.) Teacher, Monterey City Schools, California

REV. PEARSE P. DONOVAN, M.A. (San Francisco) Vice Principal, Bishop O'Dowd High School, Oakland, California

STANLEY S. DOTTERER, B.S. (Elizabethtown Coll.) Teacher, Elizabethtown High School, Pennsylvania

ALBERT DOU, D.Sc. (Madrid) Prof., University of Madrid

EDWARD DUBOIS, Student, St. Francis Xavier University

MEYER DWASS, Ph.D. (North Carolina) Asso. Prof., Northwestern University

MERLE H. EASTMAN, Accountant, Euclid 19th Auto Service, Inc., Cleveland, Ohio

GEORGE EILERS, Student, University of Santa Clara

MURRAY EISENBERG, Student, University of Pennsylvania

GEORGE R. ELLISON, B.S.E. (Oklahoma) Physics Research Asst., University of Oklahoma

RONALD J. ENSEY, Student, Hardin-Simmons University

HENRY F. EPSTEIN, Student, Bronx High School of Science, Brooklyn, New York

FLOYD A. ERICKSON, JR., Student, University of Nebraska

DAVID R. FALCONER, B.A. (Texas) Programmer, I.B.M., Bethesda

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BURTON FEIN, Student, Polytechnic Institute of Brooklyn

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PETER MENEGUS, Student, Fairfield University
R. ADM. WILLIAM L. MESSMER, M.Ed. (Duke) USN Retired; Instr., College of William and Mary, Norfolk
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LINDSEY S. PERKINS, Student, Central Michigan University
GIRARD C. PESSIS, Student, Bronx High School of Science, New York
JEAN D. PETERSEN, Student, University of Colorado

TED E. PETRIE, Student, Michigan State University

MRS. MARTHA H. PORTER, B.S. (North Texas S.C.) Grad. Student, Emory University

SIYA RAM, M.Sc. (Lucknow, India) Grad. & Teaching Asst. Statistical Laboratory, Purdue University

VICTOR L. RANSOM, M.S.E.E. (Case Inst.) Member, Tech. Staff, Bell Telephone Laboratories

ALVAH RAYMOND, M.Ed. (Boston University) Head of Dept., Weymouth High School, Massachusetts

MARY E. REED, M.A. (Nebraska) Teacher, Benton Harbor High School, Michigan

HARVEY H. RHODES, B.S.I.E. (North Dakota) System Engr., General Electric Company

JOSEPH A. RICH, Ph.D. (Yale) Physicist, General Electric Company

PATRICK RINALDO, B.B.A. (Boston) Head of Dept., Revere High School, Massachusetts

ELIODORO G. RIVERA, B.A. (Texas) Mathematician, Aerospace Medical Center

DEREK J. ROEK, M.A. (Detroit) Teaching Fellow, University of Detroit

DICK R. ROGERS, B.S. (Utah S.U.) Part-Time Instr., Utah State University

BERNHARD W. ROMBERG, B.S. (M.I.T.) Grad. Student, University of Wisconsin

PAUL ROSS, B.S. (Eng. Coll., Leningrad) Wells-Gardner Electronics Corp.

JUDITH K. ROTH, Student, Carleton College

WILLIAM G. ROUGHEAD, M.S.Ed. (Ill. State Normal U.) Instr., Northern Illinois University

WILLIAM C. SALLEE, Student, duPont Manual High School, Louisville, Kentucky

NORMAN P. SALZ, M.E.E. (Polytech. Inst. Brooklyn) Principal Engr., Cornell Aeronautical Laboratory

PETER T. SAUNDERS, Student, University of Toronto

JAMES A. SAWYER, Jr., B.S. (Stetson Univ.) Grad. Asst., Louisiana State University

ARTHUR J. SCAVELLA, M.S. (Michigan) Asst. Prof., Tuskegee Institute

LEO J. SCHNEIDER, Student, Xavier University

MRS. CERES M. SCHROER, B.A. (California) Grad. Asst., University of Rochester

LAURENCE L. SCHUMAKER, Student, South Dakota School of Mines and Technology

JONATHAN A. SEAMAN, Student, Case Institute of Technology

E. GUY SELLERS, JR., M.Ed. (Florida) Asst. Prof., Florida Southern College

ARMAND SERI, M.S. (U. of Wichita) Grad. Fellow, University of Wichita

HERBERT SHANK, M.S. (Chicago) Mathematician, Laboratories for Applied Sciences, University of Chicago

L. GORDON SHILLING, M.A. (Texas Christian) Instr., Arlington State College

NORMAN SHUSTERMAN, Student, College of the City of New York

HERBERT SICHEL, B.S. (Long Island) Senior Statistical Asst., American Cancer Society

SISTER JAMES FRANCIS, M.A. (Catholic Univ.) Instr., D'Youville College

SISTER M. ELEANOR CHRISTINE, B.S. (Villanova) Grad. Student, University of Notre Dame

SISTER MARY AQUIN, M.S. (Notre Dame) Head of Dept., Our Lady of Mercy High School, Rochester, New York

SISTER MARY BRUNO, B.A. (Mt. Mary College) Teacher, Marinette Catholic Central High School, Wisconsin

MRS. DORA SKYPEK, A.B. (Florida State Univ.) Grad. Student, Emory University

A. O. SMITH, M.S. (Wisconsin) Chairman of Dept., Brookfield High School, Wisconsin

LUCILLE A. SMITH, M.A. (Missouri) Head of Dept., Maplewood-Richmond Heights High School, Missouri

MORTON J. SMITH, Student, College of the City of New York

MITCHELL SNYDER, Student, Yeshiva University

THOMAS T. SPENGLER, Technician, Engineering Experiment Station, Georgia Institute of Technology

ROBERT SPIVACK, Student, Rutgers University

DENNIS D. SPYROS, Student, Stevens Institute of Technology

MARTIN R. STEFFEN, B.A. (Long Beach S.C.) Computer Engr., Aerojet General Corp.

DONALD J. STENGER, M.Ed. (Xavier Univ.) Teacher, Springmyer School

DANIEL J. STERLING, A.M. (Columbia) Teaching Asst., University of Wisconsin

BRYAN W. STONESTREET, M.A. (West Virginia) Instr., General Motors Institute

JOHN P. STORTO, Student, University of Detroit

EDGAR L. STOUT, Student, Oregon State College

ROBERT S. STRICHARTZ, Student, Bronx High School of Science, New York

H. E. W. NICHOLAS STURM, M.S. (Purdue) Head of Science Dept., Southwest Texas Junior College

ALVIN SWIMMER, M.S. (N.Y.U.) Teaching Asst., University of California, Berkeley

GEORGE H. TAYLOR, M.A. (Ohio S.U.) Teacher, Worthington High School, Ohio

REV. E. J. TEMPLETON, Th.D. (Philadelphia Divinity School) Chairman of Dept., St. John Baptist School, Mendham, New Jersey

GEORGE THOMAS, Sr., M.S. (Oklahoma S.U.) Asst. Prof., Shaw University

ROBERT S. D. THOMAS, Student, Leaside High School, Toronto, Canada

GUY B. TORCHINELLI, M.S. (Illinois) Asst. Prof., State University of New York, College of Education at Buffalo

EILEEN T. TORMEY, A.M. (Boston) Asst. Prof., Rhode Island College of Education

MRS. JOAN U. TREKELL, Student, University of Tulsa

JOSEPH W. TRIGG, JR., B.S. (Lincoln) Grad. Student, University of Missouri

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LEWIS E. VOGLER, B.S. (Oregon) Mathematician, National Bureau of Standards

RAYMOND E. WALCH, M.A. (Trinity) Director of Mathematics, Town of Westport, Connecticut

BEYRL E. WALROD, A.B. (Columbia) Inventory Control Supervisor, The Budd Company

MRS. HAZEL WALTERS, M.A. (Oklahoma State) Teacher, Cordell High School, Oklahoma

BENJAMIN WEISS, Student, Yeshiva University

SHIRLEY G. WEISSMAN, M.S. (C.C.N.Y.) Teacher, Hackett Junior High School, Albany, New York

WILLES L. WERNER, B.A. (UCLA) Grad. Teaching Asst., University of Utah

MACK L. WHITAKER, M.S. (Appalachian) Grad. Student, Florida State University

MILO E. WHITSON, Ed.D. (Southern Calif.) Head of Dept., California State Polytechnical College

ROBERT E. WILKERSON, B.S. (Colorado) Physicist, National Bureau of Standards

DONALD R. WILLIAMS, B.S. (Oklahoma S.U.) Grad. Asst., Oklahoma State University

RICHARD K. WILLIAMS, Student, Vanderbilt University

CARROLL K. WILSON, M.A. (N. Texas S.T.C.) Acting Chairman, Eastern New Mexico University

JAMES R. WILTS, Ph.D. (Calif. Inst. of Tech.) Asso. Math., IBM, Johnson City

ALBERT K. WONG, B.S. (Nat'l Taiwan) Grad. Student, Atlanta University

ROBERT L. WOOD, Student, St. Bonaventure University

JOHN E. WOODWARD, M.A. (Oklahoma) Mathematician, Dept. of Army, Fort Sill, Oklahoma

RAYMOND A. ZACHARY, JR., M.A. (Texas) Operations Research Analyst, Texas Instruments

RAYMOND S. ZAWADA, B.S. (Indiana Univ.) Computer Programmer, Inland Steel Company

FRANK W. ZURCHER, JR., Student, Stevens Institute of Technology

NEW SECTIONAL GOVERNORS OF THE ASSOCIATION

The following have been elected Governors of the Association for a three-year term beginning July 1, 1960 by a mail vote of the membership of the Association in the Sections indicated:

Allegheny Mountain	A. B. CUNNINGHAM, West Virginia University
Indiana	M. E. SHANKS, Purdue University
Kentucky	W. C. ROYSTER, University of Kentucky
Met. New York	C. T. SALKIND, Polytechnic Institute of Brooklyn
Nebraska	EDWIN HALFAR, University of Nebraska
Northern California	C. D. OLDS, San Jose State College
Oklahoma	O. H. HAMILTON, Oklahoma State University
Rocky Mountain	M. L. MADISON, Colorado State University
Wisconsin	R. D. WAGNER, University of Wisconsin

In four sections, more than 50% of the membership voted. Nebraska was high with a very fine 65% response. The next three were Oklahoma—57%, Kentucky—55%, and Wisconsin—51%.

L. J. MONTZINGO, *Associate Secretary*

THE TWENTY-FIRST ANNUAL WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

The twenty-first annual William Lowell Putnam Mathematical Competition will be held on Saturday, December 3, 1960. This competition, made possible by the trustees of the William Lowell Putnam Intercollegiate Memorial Fund left by Mrs. Putnam in memory of her husband, is under the sponsorship of the Mathematical Association of America and is open to regularly enrolled undergraduate students in universities and colleges of the United States and Canada who have not yet received a college degree.

Application blanks will be mailed about October first to the regular mailing list. If an application blank is not received by October 15, you may secure one by writing the director, Professor L. E. Bush, 301 Merrill Hall, Kent State University, Kent, Ohio. Your application must be filed with the director not later than November 7, 1960. For further details of the examination and the list of prizes (including the $3000 scholarship to Harvard), see the announcement which will accompany the application blank.

Reports of the previous competitions and the examinations will be found in this MONTHLY for May 1938, 1939, 1940, 1941, 1942; October 1946; August–September 1947; December 1948; August–September 1949, 1950, 1951; October 1952, 1953, 1954, 1955; December 1956; August–September (announcement of winners) and November (questions and solutions) 1957; August–September 1958; and August–September 1959.

CONFERENCES FOR MATHEMATICS INSTRUCTORS IN NSF 1960 SUMMER INSTITUTES

The Committee on Institutes of the MAA organized five regional conferences for mathematics instructors in NSF 1960 summer institutes. These conferences were financed by a grant of $49,000 to the MAA from the National Science Foundation, and were held at the following places and times: San Francisco, April 8 and 9; Washington, D. C., April 8 and 9; Chicago, April 9 and 10; New York, April 29 and 30; St. Louis, May 6 and 7.

The programs of the conferences were devoted primarily to a consideration of appropriate courses for institutes for high school mathematics teachers. Experienced lecturers outlined possible courses in algebra, geometry, analysis, probability and statistics, and foundations. These talks served as a basis for small group discussions of courses in the various branches mentioned. Reports on the group discussions were made to the entire conference in each case, with time allowed for general discussion. A talk by a high

school teacher who had attended a summer institute and a discussion of high school mathematics curricula developments by a speaker competent in this area were a part of the program of each conference. About 320 persons attended the conferences.

Summaries of the talks by the speakers and reports of the group discussions for all five of the conferences have been mimeographed for distribution. This material will be of special interest to directors and lecturers in mathematics at summer, inservice, and academic year institutes for high school teachers. Copies can be obtained free by writing to Professor H. M. Gehman, Mathematical Association of America, University of Buffalo, Buffalo 14, N. Y.

THE MARCH MEETING OF THE MICHIGAN SECTION

The annual meeting of the Michigan Section of the Mathematical Association of America was held on March 26, 1960 at the University of Michigan, Ann Arbor, in conjunction with the meeting of the Michigan Academy of Science, Arts, and Letters. Professor W. D. Baten of Michigan State University presided at both the morning and afternoon meetings and at the luncheon-business meeting. A total of 88 persons attended the meeting, including 67 members of the Association.

A nominating committee consisting of Professors F. A. Beeler, Chairman, E. E. Ingalls and B. M. Stewart proposed the following slate of officers: Professor E. D. Rainville of the University of Michigan as Chairman, Professor Wallace Givens of Wayne State University as Vice-Chairman, and Professor L. E. Mehlenbacher of the University of Detroit as Secretary-Treasurer. The slate was elected unanimously.

Professor R. M. Thrall, member of the Board of Governors representing the Michigan Section, gave a brief report of the state of the Mathematical Association of America.

Professor F. L. Celauro, Secretary of The Committee on the Michigan Mathematics Prize Competition, gave the following report:

> The Competition Examination was given on March 3, 1960 to 9200 high school contestants from 432 high schools in Michigan. The Examination consisted of two parts: Part I; objective type, designed to test general proficiency in the four years of mathematics taught in high school, and Part II; written or essay type, designed to reveal mathematical maturity, originality and clarity of expression. The awards will include 94 college scholarships valued from $120.00 to $600.00 each, six medals from the Competition Committee and 15 awards from $100.00 to $200.00.

The following motions pertaining to the Competition Committee were passed at the business meeting: (1) That the Committee for the Michigan Mathematics Prize Competition consist of six members to be appointed by the Chairman of the Michigan Section to serve terms of three years, with two new members being appointed each year; and (2) that members of the Committee be authorized to receive compensation for expenses of attending committee meetings from committee funds, depending upon the judgment of the Committee.

The following papers were presented:

1. *What statistics is, and how statisticians should be trained*, by Professor Herman Rubin, Michigan State University. (By invitation.)

We can define statistics as the theory of decision making under uncertainty. There are two kinds of uncertainty—first, an uncertainty due to probability, and second, due to the lack of knowledge of the probabilities involved. The treatment of the first is merely a problem in analysis; the second is statistics proper. Recommendations concerning the teaching of statistics are given.

2. *Inequalities related to convex curves*, by Professor J. L. Ullman, University of Michigan.

The main result of this talk is that

$$\left| \int_0^{2\pi} f(\theta) e^{-i\theta} d\theta \right| \geqq 4,$$

where $f(\theta)$ is a sense-preserving and arc-length preserving homeomorphism of the unit circle and a convex curve. This result can be interpreted to yield an interesting geometrical theorem on convex curves.

3. *Elementary covering problems*, by Professor L. M. Kelly, Michigan State University.

THEOREM. *If each three points of a set in E_n can be covered by a given closed circular disc, then the entire set can be covered by this disc.*

This theorem is due to C. V. Robinson and is a prototype of the problems with which we are concerned. For the most part this is a report of Robinson's work but a few new results are also included.

4. *Bernoulli numbers modulo 27000*, by Professor J. S. Frame, Michigan State University. This paper will be published in this MONTHLY.

5. *Emphasis on the seventh ratio in trigonometry*, by Professor G. E. Moore, Radiation Laboratory, University of Michigan.

Starting with $\theta = s/r$ and a minimum of ideas from plane geometry, practically all of the topics in trigonometry are developed and correlated. Some of the topics considered are: graphs of the functions, vectors, complex numbers and de Moivre's formula, the wrapping process and ordered number pairs, circular and inverse circular functions, function defined by number pairs, polar coordinates, roots of unity, etc. (This is somewhat the reverse of the order employed in the Report of the Commission on Mathematics—*Appendices*, and has certain advantages).

6. *Summer programs for gifted high school students*, by Professor Isobel Blythe, Michigan State University.

The National Science Foundation has supported or partially supported programs of various sorts for students who have not yet reached the college level. In 1959 there were 112 such programs in 105 institutions. Some were limited to only one subject matter area while others included several areas. The variations in length of time, facilities, purposes, staff, age of students, etc. were presented. Also the results of these programs and reactions to them by students and others were discussed. Finally, the programs for 1960, especially any significant changes, were presented.

7. *Two nonparametric significance tests*, by Professor F. A. Beeler, Western Michigan University.

The object of this talk is to present two nonparametric tests that have proved to be useful. A nonparametric significance test is a test which is valid regardless of the population from which the samples come provided that the observations are random and independent.

8. *The maximum rate in an add-on installment contract*, by Professor H. E. Stelson, Michigan State University.

In an installment contract in which the time price differential is an add-on, the yield return increases as the contract gets longer up to a point. Thereafter the yield declines as the length of the contract increases. The formula for the present value of an annuity can be differentiated implicitly but it is impossible to obtain exact maximum values for the time and rate. Approximate values of the maximum interest rate can be obtained by using approximate values of the interest rate as presented in the December 1953 issue of this MONTHLY by the author.

L. E. MEHLENBACHER, *Secretary*

THE APRIL MEETING OF THE ALLEGHENY MOUNTAIN SECTION

The 34th meeting of the Allegheny Mountain Section of the Mathematical Association of America was held at Grove City College, Grove City, Pennsylvania, on April 30, 1960. Professor P. N. Carpenter of Grove City College presided at the morning session,

and Dr. B. H. Mount, Chairman of the Section, presided at the afternoon session. Thirty-eight persons were registered for the meeting.

The morning session was devoted to technical papers, as reported below. During the afternoon session, a report was made on the high school examinations in West Virginia and in western Pennsylvania. Income from the contest examinations has exceeded expenses over the past three years. Numerous suggestions were offered for the best use of these funds in improving the program of high school examinations. Final action was deferred until the next meeting.

Dr. B. H. Mount read a paper prepared by Professor J. H. Neelley on the "Joint MAA and NCTM Meeting in Chicago on January 30," in which the author reported his impressions of the meeting.

Professor H. F. Bechtell presented a "Progress Report on the Grove City Plan." Grove City College installed a mathematics curriculum that requires all students taking elementary mathematics through the calculus to follow the same sequence of courses and terminating them at the end of the sophomore year. (Cf. May Meeting of the Allegheny Mountain Section, this MONTHLY, volume 66, number 7.) The plan has very successfully met its objectives. A survey analysis of the prerequisites of the entering students and their eventual success in the first course of algebra and trigonometry strongly supports it as the initial course in the sequence.

Professor J. C. Knipp distributed copies of and discussed a "Report on the Mathematics Program of the Regional Commission on Educational Coordination." Recommendations for coordination of mathematics curricula in high schools and universities were prepared by a panel of mathematics teachers and industrial mathematicians from the western Pennsylvania area.

The following papers were presented:

1. *Maclaurin coefficients for trigonometric and hyperbolic functions*, by Professor F. H. Steen, Allegheny College.

The recursion formula

$$A_n = B_n + 4\binom{n}{4}A_{n-4} - 4^2\binom{n}{8}A_{n-8} + 4^3\binom{n}{12}A_{n-12} - \cdots$$

in which A_n represents the nth derivative of such functions as sec x, sech x, tan x, evaluated at $x=0$, and B_n represents the corresponding values of the product of the function by cos x cosh x, was derived and discussed, together with a similar formula for functions such as x csc x, x csch x, x^2 csc x csch x, x cot x.

2. *Normal approximation to the distribution of two independent binomials, conditional on fixed sum*, by Professor W. L. Harkness, Pennsylvania State University.

Using Feller's superior normal approximation theorem for the binomial distribution, it is shown under suitable restrictions that the distribution of two independent binomials is, for fixed sum, asymptotically normal. The result has applications in the study of asymptotic power functions for the test of independence in 2×2 contingency tables.

3. *A binomial series transformation*, by Mr. H. W. Gould, West Virginia University.

Let $f(k)$ be independent of n and define

$$F(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{a+bk}{n} f(k), \qquad f(0) = 1.$$

Then

$$\sum_{k=0}^{\infty}\binom{a+bk}{k} z^k f(k) = x^a \sum_{n=0}^{\infty} (x^{-1}-1)^n F(n),$$

where $z=(x-1)x^{-b}$. This generalizes a procedure used by the author (this MONTHLY, vol. 63, 1956, pp. 84–91) to obtain sums of certain infinite series.

4. *Some relations between alternative rational-choice criteria for consumer behavior theories based on weak orderings*, by Professor E. M. Fels, University of Pittsburgh, introduced by the Secretary.

Starting from formalizations of choice-set functions (as generalized consumer-demand functions), weak (i.e. connected and transitive) orderings, and revealed and indirectly revealed preference relations, five alternative criteria for "rational" choice functions (by Ville, Houthakker, Arrow, Chernoff, and Samuelson) are investigated with a view to their mutual derivability. A conjecture that the so-called Weak Axiom of Revealed Preference (Samuelson) implies the so-called Strong Axiom of Revealed Preference (Houthakker, Ville) is refuted (unless some definitions are changed), utilizing a systematic refutation procedure in accordance with Quine's (1955) proof procedure for quantification theory.

EVAN JOHNSON, JR., *Secretary*

THE APRIL MEETING OF THE IOWA SECTION

The 47th regular meeting of the Iowa Section of the Mathematical Association of America was held at the State University of Iowa, Iowa City, on the afternoon of April 22 and the morning of April 23, 1960. Professor J. J. L. Hinrichsen, Chairman of the Section, presided. Total attendance was 115, including 45 members of the Association.

The following officers were elected: Chairman, Professor H. C. Trimble, Iowa State Teachers College; Vice-Chairman, Reverend T. J. Taylor, St. Ambrose College; Secretary-Treasurer, Professor E. L. Canfield, Drake University.

The following papers completed the program:

1. *Approximation on an infinite interval*, by Professor E. N. Oberg, State University of Iowa.

By means of the criteria of least mth power methods, and an integral formulation of Bernstein's theorem, it is shown that a function $f(x)$ that is continuous and which is of the order $c_0x^k+c_1x^{k-1}+\cdots+c_n$ as x becomes infinite can be approximated by sums of orthonormal functions whose derivatives are orthogonal. The rapidity of convergence can be shown to depend upon the properties of continuity of the function $f(x)$.

2. *Explicit formulas for some continuous probability measures*, by Professor C. E. Langenhop, Iowa State University.

For any real x, $0\leq x\leq 1$, $x=\sum_{i=1}^{\infty} a_i2^{-i}$, where for each i, $a_i=1$ or 0. Let $S_n=\sum_{i=1}^{n} a_i$ and define $F(x)=\sum_{i=1}^{\infty} a_iq^i(p/q)^{S_{i-1}}$, where $p+q=1$, $p>0$, $q>0$, It can be shown that F is a continuous monotone increasing function on $0\leq x\leq 1$ with $F(0)=0$ and $F(1)=1$ and that F is the distribution function on $0\leq x\leq 1$ induced by the probabilities of finite sequences of Bernoulli trials with probabilities of success ($a_i=1$) and failure ($a_i=0$) on each trial being p and q, respectively.

3. *The two-color theorem*, by Professor L. E. Pursell, Grinnell College.

The "two-color" theorem asserts that if a sphere is separated into regions by a finite graph all of whose vertices are of even order, then it is possible to partition this collection of regions into two classes (or colors) such that no two regions of the same class have an arc of the graph in common. This theorem has been stated in the literature, but the author does not know of the publication of a proof of the theorem (except for special cases). Two proofs of this theorem are given. With proper restrictions on the graph, these proofs are valid for any surface which satisfies the Jordan Curve Theorem.

4. *A revision of a completion method and its adaptation to ill-conditioned matrices*, by Professor W. L. Waltmann, Wartburg College.

An essentially new method for the inversion of $n\times n$ matrices which is closely related to the method of completion attributed to Sherman, Morrison, and Bartlett has been developed. Their method gives the inverse for a nonsingular matrix that differs in only one column from a matrix whose inverse is known. The revision begins with a diagonal matrix and by successive column

changes produces the inverse of the desired matrix. This technique controls the conditioning of the matrix by controlling both the size of its elements and its determinant. The revised method can be readily adapted to electronic digital computers.

5. *Newtonian and Galilean reference frames (of the special theory of relativity) defined and compared by elementary mathematics*, by Professor C. J. Wolfsa, Grinnell College, introduced by the Chairman.

A survey is made of the available literature regarding the terms, reference frame, reference system, Galilean system, Newtonian system, and inertial systems. Precise mathematical definitions of these terms are constructed culminating in the definition of a Galilean inertial reference system. A survey is made concerning the linearity premise used by Einstein, and many others after him, in deriving the Lorentz transformation equations for special relativity. Lastly, this linearity premise is shown to follow directly from the previous definition of a Galilean inertial reference systems.

6. *On linear combinations of unbiased statistics*, by Professor R. V. Hogg, State University of Iowa.

Let T_1 and T_2 be independent unbiased statistics for the paramenter m. Let σ_1^2 and σ_2^2 be respectively the variances of these statistics. It is well known that $T=(\sigma_2^2T_1+\sigma_1^2T_2)/(\sigma_1^2+\sigma_2^2)$ is the unbiased linear combination of T_1 and T_2 having smallest variance. If σ_1^2 and σ_2^2 are unknown, we define new statistics T' by replacing, in T, σ_1^2 and σ_2^2 by various estimates of them. It is proved that if m is the midpoint of a symmetric distribution, if T_1 and T_2 are unbiased odd location statistics for m, and if the estimates of σ_1^2 and σ_2^2 are even location-free statistics, then each T' is an unbiased statistic for m.

7. *Generalized Riemann integrals in subspaces of the reals*, by Mr. T. J. Robinson, University of North Dakota, and Professor D. E. Sanderson, Iowa State University, presented by Mr. Robinson.

For real-valued functions defined on $E\subset[a, b]$, the integral as a limit of "Riemann" sums, upper sums S, and lower sums s, for partitions of norm Δ, are defined as usual except only subintervals containing points of E are used. The upper (lower) integral is $J=\sup_{\Delta>0}\inf_{\Delta<\delta} S$ ($I=\inf_{\Delta>0}\sup_{\Delta<\delta} s$). Results: $\lim_{\Delta\to 0} S=J$, $\lim_{\Delta\to 0} s=I$ and the integral exists if and only if (i) $I=J$, (ii) given $\epsilon>0$, $S-s<\epsilon$ for some partition, or (iii) $\lim_{x\to a}\inf f(x)=\lim_{x\to a}\sup f(x)$ for almost all a in E. Functions integrable on E are extendable to integrable functions $\bar{E}([a, b])$ whose integral over all subsets is the same (arbitrarily close—equality not always attainable).

8. *A remark on a certain quadratic form*, by Professor A. T. Craig, State University of Iowa.

Let Q_1 denote a real symmetric quadratic form of rank r, $0<r<n$, in n random values of a variable that is normally distributed with mean μ and variance σ^2; and let the distribution of Q_1 be free of μ. Let Q_2, stochastically independent of Q_1, be another quadratic form in the same random variables. If the rank of Q_2 is $n-r$, the distribution of Q_2 must depend upon μ.

9. *Methods of nomographic disjunction*, by Joanna Enzmann, Sylvania Electronic Systems, introduced by the Chairman.

In order to construct a nomogram from a function of three variables, the function must first be put into the proper form. This form is a three by three determinant, called the disjunction, each row of which contains only one variable, and which, when expanded, yields the original function. To do this, one must find a minimal basis for one variable, and, using this basis as the first row of the determinant, find the other two rows by numerical means. Certain special and degenerated cases may be solved by modifying the general method, and in addition some special cases may be solved by using a partially graphical method.

10. *A theoretical study of a randomly excited mechanical oscillator*, by Professors V. W. Bolie and F. M. Long, Iowa State University, presented by Professor Bolie, introduced by the Chairman.

The paper deals with a mathematical investigation of the feasibility of extracting usable mechano-electric power from a critically damped and randomly excited mechanical oscillator.

Two different analytic functions are used to approximate practical squared-velocity spectra for the motion of the vehicle supporting the oscillator, and the corresponding expressions for extractable power are developed. Numerically computed results are presented graphically to illustrate the effects of varying the oscillator parameters.

11. *Error terms of numerical integration formulas*, by Professor R. J. Lambert, Iowa State University.

This paper gives a proof that the polynomial function $Q_n(s)=\int_0^s t(t-1)\cdots(t-n)dt$ does not change sign on the interval $(0, n)$. Heretofore it was generally believed that $Q_n(s)$ changed sign on the interval $(0, n)$ when n was an odd integer. The technique of proof is to show that when n is odd, $Q_n(s)$ has an upper bound $Q_n(n-1)$ for $(n-1)/2 \leqq s \leqq n$ which is shown to be negative. This result simplifies the treatment of the error terms in certain numerical integration formulas which involve divided differences. The simplified treatment is given here.

12. *A note on dialectics in mathematics*, by Professor W. A. Small, Grinnell College.

By the use of some elementary set-theoretic concepts, a precise formalization and a generalization are given of a result stated by Frederick Engels on the meaning of dialectics in mathematics. A complete dialectical process is defined, and it is shown that such a process is a function, and that every real function is a complete dialectical process. Some general implications of this result are discussed.

13. *Convex functions*, by Professor W. T. Reid, State University of Iowa (by invitation).

This paper is devoted to a survey of basic properties of functions which are convex on an open interval, and extensions of these properties to the case of functions that are convex relative to the totality of solutions of a nonoscillatory linear generalized differential equation $d[r(t)u'(t)] - u(t)dm(t)=0$. Special attention is given to those properties that are variational in nature.

14. *Homomorphisms of extensive systems*, by Mr. M. F. Ruchte, Ames, Iowa, introduced by the Chairman.

15. *Logical and semantical examination of non-Euclidean geometries*, by Professor H. G. Apostle, Grinnell College.

Intuition is necessary in mathematics. If "space" and "dimension" are restricted to physical space, a univocal use of "straight line" leads to logical difficulties, and an ambiguous use does not necessarily refute Kant or render Euclidean geometry a special case of geometry. Further, interpretations of non-Euclidean geometries by recourse to Euclidean models indicate that non-Euclidean geometries may be branches of Euclidean geometry. On the other hand, the treatment of geometry from the general to the particular (Von Staudt, Klein, Veblen, and Young, etc.) is scientifically better from the modern as well as from the ancient (Plato, Aristotle) point of view.

E. L. CANFIELD, *Secretary*

THE APRIL MEETING OF THE KANSAS SECTION

The forty-fifth annual meeting of the Kansas Section of the Mathematical Association of America was held at Kansas State College of Pittsburg, Pittsburg, Kansas, on April 30, 1960, in conjunction with the annual meeting of the Kansas Association of Teachers of Mathematics. Of the 175 people registered, 69 were members of the Association. Professor J. D. Haggard, chairman, presided at the sessions.

The following officers were elected for one-year terms: Chairman, Professor W. D. Bemmels, Ottawa University; Vice-Chairman, Professor L. E. Fuller, Kansas State University; Secretary-Treasurer, Miss Helen Kriegsman, Kansas State College of Pittsburg.

At the joint session, held in the morning, Professor M. R. Hestenes, University of California, Los Angeles, addressed the group on *Some properties of matrices*.

The following short papers were presented at the afternoon session:

1. *Affine spaces*, by Professor T. J. Head, Washburn University.

A relation between the theorem of Cayley in group theory and the concepts of parallelism and translation was discussed.

2. *The teaching of mathematics at British universities*, by Professor Wendy J. Robertson, University of Kansas.

The mathematical training offered in British schools and universities differs in several respects from that available here; some of these differences were discussed. Examples were given of what sort of mathematics a student might have learned in typical English and Scottish schools. A description followed of the methods of teaching and of the examination system in various universities. The contents of courses leading to Pass degrees in subjects including mathematics and to Honours degrees in mathematics, and those given to students in engineering and science, were summarized in detail.

3. *Some uses of matrices in coding*, by Professor L. E. Fuller, Kansas State University.

Messages can be coded by setting up numerical equivalents for the letters of the alphabet, two marks of punctuation, and one for a space. The message is divided into sets with the same number of symbols. Column vectors are formed from the numerical equivalents and multiplied by a square matrix over I_{29}. The resulting vectors are the numerical equivalents of the coded message. The same process using the inverse of the matrix will decode any message. The numerical part of the process is readily adapted to use of a desk calculator. It can also be programmed for a digital computer.

4. *Solvability of factorizable groups*, by Professor L. E. Laird, Kansas State Teachers College.

The classification of the subgroups, factor-groups, and direct products of cyclic, Abelian, nilpotent, supersolvable, and solvable finite groups is well known. The problem of classification of a group $G = HK$, where the classifications of the proper subgroups H and K are known, has received attention in the past ten years. The present status of the problem was discussed by the speaker.

HELEN KRIEGSMAN, *Secretary*

THE APRIL MEETING OF THE MISSOURI SECTION

The Missouri Section of the Mathematical Association of America met at Central Missouri State College, Warrensburg, Missouri, on Saturday, April 30, 1960. Professor L. O. Jones, Vice-Chairman of the Section, presided at the morning session and Professor C. E. Kelley, Chairman of the Section, presided at the afternoon session. Forty-four persons, including 31 association members, attended these meetings.

The officers for 1960–1961 are: Chairman, Professor J. L. Zemmer, University of Missouri; Vice-Chairman, Professor J. J. Andrews, St. Louis University; Secretary-Treasurer, Professor N. A. Haynes, University of Missouri. Mr. R. L. Spreckelmeyer, retiring chairman of the Committee on the High School Mathematics Contest, gave a report on the progress made in this project. As a result of discussion introduced by Professor P. B. Burcham, University of Missouri, a committee is to be appointed to investigate the formation and administration of a test for placement and/or credit to be used by the schools in the state.

The program for the meeting consisted of two parts, one in conjunction with the morning session and one at the afternoon session. The following papers were given:

1. *Euclidean geometry without postulates*, by Professor L. M. Blumenthal, University of Missouri. (By invitation)

Distinguishing between "postulates" (statements in the primitive notions *peculiar* to a subject) and "axioms" (common notions), plane Euclidean geometry is based here on the sole common notion "real number" in a manner to exhibit every Euclidean invariant in terms of the mutual

distances of finite subsets of points. Let $(12)=(21)$, $(23)=(32)$, $(13)=(31)$ denote three positive numbers whose Cayley-Menger determinant $D(1, 2, 3)<0$. A point r is defined as an ordered triple of nonnegative real numbers $(1r, 2r, 3r)$ such that $D(1, 2, 3, r)=0$. Distance of points r, s is the unique nonnegative real root of the equation $D(1, 2, 3, r, s; x)=0$, obtained by replacing elements rs and sr in $D(1, 2, 4, r, s)$ by x.

2. *Reckoning and reasoning*, by Professor R. C. Fisher, The Ohio State University. (By invitation)

Axiomatics enters mathematics in two different ways, for different purposes. A complete set of axioms for a mathematical system such as the real number system or Euclidean geometry is given for the purpose of completely specifying the interpretation of that system, that is, it is restrictive in purpose. On the other hand, certain mathematical systems (such as groups) are given axiomatic treatment for purposes of identifying a type of structure in several mathematical systems. High school teachers might better emphasize the latter role of axiomatics (as it is related to structure) than to engage in too formal an approach to mathematics from logical foundations.

MARIAN A. LESHER, *Secretary*

THE APRIL MEETING OF THE NEBRASKA SECTION

The thirty-sixth annual meeting of the Nebraska Section of the Mathematical Association of America was held on April 22–23, 1960, at the University of Nebraska, Lincoln, Nebraska, in conjunction with the seventieth annual meeting of the Nebraska Academy of Sciences. Professor D. W. Miller, Chairman of the Section, presided. There were 50 persons present at the sessions, including 35 members of the Association.

The following officers were elected for 1960–61: Chairman, Professor R. L. Moenter, Midland College; Vice-Chairman, Professor D. W. Miller, University of Nebraska; Secretary-Treasurer, Professor H. M. Cox, University of Nebraska.

Professor J. M. Earl was continued as Chairman of the Committee on Mathematics Contests. The Committee consists of representatives of the Nebraska Section of the Mathematical Association of America, the Nebraska Section of the National Council of Teachers of Mathematics, the Nebraska Actuaries Club, and the Nebraska Academy of Sciences.

Professor L. J. Mordell of Cambridge University, Visiting Professor at the University of Colorado, gave the Academy Address on the subject *Reflections of a Mathematician.*

At the Association meetings, the following papers were presented:

1. *Recent results in number theory*, by Professor L. J. Mordell. (By invitation)

2. *Subgroups of division rings*, by Professor William Scott, University of Kansas. (By invitation)

3. *The undergraduate mathematics curriculum in Nebraska colleges and universities*, Panel discussion, by Professor Joseph Wampler, Nebraska Wesleyan University; Professor L. M. Larsen, Nebraska State Teachers College, Kearney; Professor A. K. Bettinger, Creighton University; Professor J. M. Earl, the University of Omaha; Professor Hubert Hunzeker, University of Nebraska, Moderator.

4. *Results of the Nebraska Mathematics Contest*, by Professors J. M. Earl, University of Omaha, and H. M. Cox, University of Nebraska.

The 1960 Mathematics Contest had 2,616 students from 140 Nebraska High Schools enrolled. This compares with 1,635 and 2,428 students from 127 and 133 schools in 1958 and 1959. Of the total of 211 different schools, 63, 62, and 86 were entered in 3, 2, and 1 contests respectively.

5. *On a necessary and sufficient condition that a ring be a field*, by Professor G. H. Meisters, University of Nebraska.

6. *Latin squares*, by Professor Trevor Evans, Emory University and the University of Nebraska. (Invited paper)

7. *Factorizable groups*, by Professor William Scott, University of Kansas. (Invited paper)

8. *Equations that have only trivial models*, by Mr. Robert Schwabauer, University of Nebraska.

As applied to groupoids, the trivial one element algebra is a model of every equation. For some equations it is the only model. Such an equation must be of the form $x=\alpha$, where x is not the first or last variable in α. A sufficient condition for $x=\alpha$ to have no nontrivial models is that no variable appear more than once in α and x is not the first or last variable in α.

9. *An elementary proof of the Schroeder-Bernstein theorem*, by Professor H. H. Schneider, University of Nebraska.

The Schroeder-Bernstein Theorem states: If A, B and C are sets such that A is a subset of B and B is a subset of C, and if there exists a one-to-one correspondence between the elements of A and of C, then there exists also a one-to-one correspondence between the elements of A and of B. The proof of this theorem is given by elementary means, that is, no use is made of the axiom of choice in any form whatsoever. No construction is employed which involves any countable process such as recursive definitions of mappings, etc., or which uses properties of indefinite sets. The proof can easily be formalized within an axiomatic treatment using only the basic set-forming axiom-schemata.

10. *Recent investigation of the Selberg sieve method*, by Professor W. E. Mientka, University of Nebraska.

This paper indicates how the lower bound of the Selberg sieve method was used to obtain two theorems concerning the problem of the existence of at least one prime between consecutive perfect squares. The problem of determining the upper and lower bounds for various sets of numbers satisfying certain conditions is also considered along with possible application to some unsolved problems in number theory.

11. *Semigroups with restricted subsemigroups*, by Mr. C. V. Heuer, University of Nebraska.

Call a semigroup $(S,.)$ a T-semigroup if each proper subsemigroup of $(S,.)$ possesses the property of T. Several results concerning the structure of such semigroups are obtained, of which the following is typical. Theorem: Let $S=(S,.)$ be a semigroup each of whose proper subsemigroups is a group. (i) If S is cyclic, with generator a, then S is finite and either S is a group or $S-(a)$ is a group. (ii) If S is not cyclic then either S is a group or S is the multiplicative semigroup of the field with two elements.

H. M. Cox, *Secretary*

THE APRIL MEETING OF THE SOUTHEASTERN SECTION

The thirty-ninth annual meeting of the Southeastern Section of the Mathematical Association of America was held April 1–2, 1960, at the Wade Hamilton Hotel, Columbia, South Carolina, with the University of South Carolina acting as host institution. Professors T. C. Carson, Chairman of the Section, T. H. Lee, Vice-Chairman, H. A. Robinson, W. L. Williams and E. B. Shanks presided over the general and divisional sessions. There were 176 in attendance, including 140 members of the Association.

The following officers were elected for the coming year: Chairman, Professor H. A. Robinson, Agnes Scott College; Vice-Chairman, Professor E. B. Shanks, Vanderbilt University; Secretary-Treasurer, Professor C. L. Seebeck, Jr., University of Alabama. Professor J. G. Kemeny, Dartmouth College, presented the featured lecture at the invitation of the Section and the University of South Carolina. At the business meeting a committee consisting of Professors H. A. Robinson, Tomlinson Fort, and C. L. Seebeck, Jr. was formed to propose a plan for splitting the Section. It was decided that a mail ballot of all members of the Section be the means of deciding on the committee's proposal.

The Section took no action on a proposal to form a committee to study participation in the High School Mathematics Contest.

The following program was presented:

1. *Bounds for positive solutions of systems of linear diophantine equations*, by Professor B. M. Seelbinder, Wake Forest College.

Consider the system of diophantine equations $\sum_{j=1}^{k} a_{ij}x_j = v_i (i=1, \cdots, n)$, where $n \leqq k$, $a_{ij} \geqq 0$, are given integers such that $\sum_{i=1}^{n} a_{ij} > 0$ for $j=1, \cdots, k$ and the n-rowed minors of the coefficient matrix A have g.c.d. unity. Let D be any positive valued n-rowed minor with elements a_{ij_r}, $r=1, \cdots, n$. The claim is made that for solvability of the system in positive integers it is sufficient that $(v_1, \cdots, v_n)$ lie in the interior of a convex polyhedral cone determined by the rays from the point $(f_1, \cdots, f_n)$ to the points $(f_1+a_{1j_r}, \cdots, f_n+a_{nj_r})$, $r=1, \cdots, n$, where

$$f_i = (D/t) \sum_{j=1, j \neq j_r}^{k} a_{ij},$$

t=g.c.d. of elements of D; r, $i=1, \cdots, n$.

2. *Variation of parameters: an explanation*, by Professor C. G. Phipps, University of Florida.

In solving differential equations, textbooks merely show *how* the method of variation of parameters works. The purpose of this paper is to show *why* it works.

3. *Generalization of two well-known results concerning Hermitian forms*, by Professor Lonnie Cross, Atlanta University, (presented by title).

This paper establishes the following two generalizations of two well-known results concerning Hermitian forms: (1) Let A and B be bounded, Hermitian integral operators defined on $L^2(a, b)$ and be such that A is completely continuous, B is strictly positive definite, $|(A\phi, \phi)| \leqq C(B\phi, \phi)$ $(C>0)$, $\phi \in L^2(a, b)$ Then the equation (*) $A\phi = \rho B\phi$ has a discrete spectrum. Moreover, if A is positive definite, then for any $\phi \in L^2(a, b)$ $(A\phi, \phi) = \sum_{r=1}^{\infty} \rho_r |(B\phi, \phi_r)|^2$, where the ρ_r are the eigenvalues of (*), ϕ_r the corresponding eigenfunctions. (2) (Minimum-maximum characterization of the eigenvalues of (*))Let $g_1, \cdots, g_{n-1}$ be elements of $L^2(a, b)$ and denote by $\rho_n(g_1, \cdots, g_{n-1})$ the maximum of $(A\phi, \phi)$ under the conditions $(B\phi, \phi)=1$ and $(B\phi, g_i)=0$ $(i=1, \cdots, n-1)$. Then the nth positive eigenvalue ρ_n^+ is equal to the minimum of $\rho_n(g_1, \cdots, g_{n-1})$ for all possible choices of the functions $g_1, \cdots, g_{n-1}$, and it is attained for $g_i = \phi_i^+$ $(i=1, \cdots, n-1)$.

4. *Matrix slide rules*, by Professor E. P. Miles, Jr., Florida State University.

This paper will appear as a Classroom Note in this MONTHLY.

5. *Research and teaching in college*, by Professor T. C. Carson, East Tennessee State College.

The speaker called attention to a growing tendency of college and university administrations to stress research rather than teaching. This increasing practice has the following very serious results: 1st. To place inexperienced fellows and scholars to do the teaching of young students while the older experienced and higher salaried men are engaged in research is cheating them of their rights as students and their parents of the money they have invested in the school and students' expenses. 2nd. The grading of faculty members by the amount of research published is a very poor standard due to varying subjects and amount of time it takes for each subject. 3rd. The emphasis on research is harmful to research itself. Failure to develop sound backgrounds in the student tends to fail to develop promising candidates for research in the future. Teaching is the most important function of a college faculty. Research should be a secondary part of its work.

6. *Some experiences in teaching advanced mathematical concepts to talented 4th–9th graders*, by Professor Robert Kalin, Florida State University.

Experiences with Florida State University Mathematics Camp, mathematics clubs, and demonstration classes suggest that bright 4th–9th graders can learn abstract, advanced mathematical

ideas, if properly presented. Subject matter taught has included equation and inequality solution, finite arithmetics, etc. Demonstration classes proved excellent for showing teachers new ideas in mathematics and its instruction.

7. *The hyperbolic functions are elementary*, by Professor C. L. Seebeck, Jr., University of Alabama.

The functions cosh ϕ and sinh ϕ are defined as the coordinates (x, y) of the curve $x^2-y^2=1$, where ϕ is the usual area. Starting with three defining postulates for logarithms, the functions $\ln x$ and e^x are developed, and the usual expressions for the hyperbolic functions in terms of e^ϕ are obtained using only high school mathematics.

8. *A matrix problem for college students*, by Professor R. C. Blackwell, Furman University.

The determinant of the square matrix formed by taking the variable in every position and adding a constant to every position of the main diagonal was shown to be the product of the constants plus the variable multiplied by the sum of the products of the constants taken $n-1$ at a time. The proof was given by altering the first two rows and applying a Laplace development and an induction. It was suggested that the evaluation might be a good exercise for students beginning the study of determinants.

9. *An application of skip-sampling to student performance*, by Professor M. C. Palmer, Clemson College.

While most of us who teach mathematics would like to direct our efforts toward our more zealous students, we often find ourselves teaching larger and larger sections in which many of the students are somewhat less than ideal. The purpose of this paper is to focus attention to the problem of producing a maximum effort from the more or less dilatory students while preserving the energy of the instructor for the more pleasant aspects of his occupation. A method of sampling students' assignments is proposed which places a large measure of pressure on the students while requiring a modest amount of paper work by the instructor.

10. *Global stability theory for a particular system of nonlinear differential equations*, by Professor J. A. Nohel and Mr. W. P. Timlake, Georgia Institute of Technology, presented by Mr. Timlake.

For the real nonlinear system $(*)\dot{x}=-a_0(y)y-\sum_{i=1}^{n}a_i(z_i)z_i$, $\dot{y}=b_0(y)f(x)$, $\dot{z}_i=-h_i(t, x, y, z_1, \cdots, z_n)z_i+b_i(z_i)f(x)$, $i=1, \cdots, n$, where a_i, b_i, h_i, f are given functions and $(\cdot=d/dt)$, sufficient conditions are given which insure that every solution of $(*)$ exists on $0\leqq t<\infty$ and approaches zero as $t\to+\infty$. This result, an extension of a theorem of Levin and Nohel (to appear Arch. Rational Mech. Anal., 1960), can also be obtained from a theorem proved independently by Levin.

11. *On criteria for nonlinear stability*, by Professor C. W. McArthur, Florida State University.

For each real number t let $z(t)$ denote an element of Euclidean n-space E_n and $A(t)$ an $n\times n$ real matrix. Let $\lambda(t)$ and $\mu(t)$ denote the least and largest eigenvalues respectively of the matrix $\frac{1}{2}[A(t)+A^T(t)]$ where A^T is the transpose of A. Let f denote a function on E_n to E_n such that $f(0)=0$. Suppose $z(t)$ is a nontrivial solution of the system $dz/dt=A(t)z+f(z)$. It follows that $\lambda(t)-\|f(z)\|/\|z\|\leqq d(\ln\|z(t)\|)/dt\leqq\mu(t)+\|f(z)\|/\|z\|$. Further, if $(\|f(z)\|/\|z\|)\to 0$ as $\|z\|\to 0$, if $\mu(t)$ is integrable, and if there is a $b>0$ such that $\mu(t)+b<0$ for all t, it is shown that the identically zero solution of the above system is asymptotically stable.

12. *A stability theorem for a system of delay differential equations*, by Mr. L. J. Grimm and Professor J. A. Nohel, Georgia Institute of Technology, presented by Mr. Grimm.

Results for a single delay differential equation by the senior author (J. Math. Phys., vol. 38, 1960, pp. 295–311) are obtained for a particular system of such equations.

13. *An analysis of the Rosser rocket function*, by Mr. W. L. Hales, University of Alabama.

The generalized Rosser rocket function and its relationships to other functions are examined. It is found to generate a series of incomplete Beta functions.

14. *Mean value theorems for functions with finite derivates*, by Professor D. B. Goodner, Florida State University.

Since the law of the mean and many of its generalizations require the existence of the derivative, it seems desirable to have expressions which may serve us when there is no derivative. This paper presents extensions of the law of the mean, Taylor's formula with remainder, and Cauchy's generalized law of the mean to functions which have finite one-sided upper and lower derivates.

15. *Differential equations of the first order and first degree*, by Professor Tomlinson Fort, Emory University.

This paper points out the advantages of treating the equation of the first order and first degree by definitions and methods of solution that are symmetric in x and y. Such a treatment avoids difficulties brought in by implicit functions. The paper emphasizes the necessity of discussing domain and range at all times. A definition of a solution is given and it is shown that many functions not usually considered solutions must properly be so described.

16. *On the representation of integers by ternary quadratic forms*, by Professor E. H. Hadlock, University of Florida.

The method employed by L. E. Dickson in finding the progressions of integers which are represented by a genus of positive ternary forms with given invariants is generalized in the form of several theorems. Applications of these theorems involve very little work in finding the progressions of integers represented by a genus of positive forms. The theorems also apply to indefinite forms.

17. *The summation of trigonometric series*, by Professor C. B. Smith, University of Florida.

It is well known that certain special types of trigonometric series may be summed by means of the theory of residues. This summation depends on the integration of an appropriate meromorphic function around the contour $|z|=R$, where R is eventually taken to be infinite. This paper gives a method for finding the sums of more general types of trigonometric series by using contour integration around the large circle $|z|=R$ and the unit circle. This procedure also gives the values of some rather unusual looking definite integrals.

18. *Markov chain theory—expository treatment*, by Professor J. G. Kemeny, Dartmouth College.

Markov chain theory is introduced through a series of simple, concrete examples. Basic theorems on both absorbing and ergodic chains are stated without proof, and illustrated in applications. It is briefly shown how the theory of finite Markov chains can be reduced to matrix problems. Some results and certain unsolved problems for infinite Markov chains are mentioned.

19. *Quasi-isomorphisms of certain algebras*, by Professor B. F. Bryant and Mr. J. F. Kepler, Vanderbilt University.

A one-to-one correspondence $x \rightarrow x'$ of a ring R onto a ring R' is called an anti-isomorphism {quasi-isomorphism} provided (1) $(a+b)'=a'+b'$ and (2) $(ab)'=b'a'$ {for each pair a and b in R, at least one of the equations $(ab)'=a'b'$, $(ab)'=b'a'$ holds}. Hua (*On the Automorphisms of an S-field*, Proc. Nat. Acad. of Sci. U.S.A., vol. 35, 1949, pp. 386–389) has shown that each quasi-isomorphism of a ring R onto a ring R' is either an isomorphism or an anti-isomorphism. For algebras with one binary operation, requirement (1) in the above definitions is dropped. Examples were given to show that (1) there do exist proper quasi-isomorphisms of loops; *i.e.*, quasi-isomorphisms which are neither isomorphisms nor anti-isomorphisms, (2) some loops are not properly quasi-isomorphic to any loop, and (3) a loop may be properly quasi-isomorphic to non-isomorphic loops, which is in contrast to the situation for anti-isomorphisms.

20. *Busche-Ramanujan identities*, by Professor P. J. McCarthy, Florida State University.

The class of arithmetical identities called "Busche-Ramanujan identities" by Vaidyanathaswamy (Trans. Amer. Math. Soc., vol. 33, 1931, pp. 579–662) are discussed, and a generalization of these identities is obtained.

21. *Subspaces spanned by discrete polynomials*, by Professor Andrew Sobczyk, University of Florida.

In any vector space of dimension $N=2^\nu$, ν any positive integer, there exists an orthogonal basis of vectors of the form $\phi^{(1)}=\{\epsilon_t^{(1)}\}, \cdots, \phi^{(N)}=\{\epsilon_t^{(N)}\}$, $t=1, \cdots, N$, where $\epsilon_t^{(k)}=\pm 1$ for all k and t. Denote the orthonormal polynomials on the set $(1, \cdots, N)$ by $p_0(t), \cdots, p_{N-1}(t)$, the degree being indicated by the subscript. The matrix of the orthogonal transformation between $\phi^{(1)}, \cdots, \phi^{(N)}$ and $p_0, \cdots, p_{N-1}$ is determined; in particular if S_k denotes the subspace containing p_k which is spanned by the minimal subset of $\phi^{(1)}, \cdots, \phi^{(N)}$, it is shown that S_0, S_1, S_2 are mutually orthogonal, of respective dimensions 1, ν, and $\frac{1}{2}\nu(\nu-1)$.

22. *On certain functions of a set and a point*, by Professor L. T. Ratner, Vanderbilt University.

This paper deals with real-valued functions $f(x, S)$, where x and S are a point and a subset, respectively, of a given topological space X. Several conditions on X, S and/or $\{f(x, S)\}$ are developed which insure the continuity of $g(S)=_{Df} f(x, S)$ from 2^X to R^X.

23. *A probabilistic inequality for testing two distribution functions*, by Professor H. P. Kuang, Agricultural and Technical College of North Carolina.

Let the discrete marginal distributions of the two stochastic variables X and Y be $F_1=\{p_i\}$ and $F_2=\{q_i\}$ $(i=1, \cdots, r)$ respectively, where $p_i=P$ $(X=s_i)$, $q_i=P$ $(Y=s_i)$; $\sum_{i=1}^r p_i=1$, $\sum_{i=1}^r q_i=1$, and s_i is an arbitrary interval containing a finite number of the mass points. Let further the corresponding observations of X and Y be $\{n_i\}$ and $\{m_i\}$ respectively, where the sample values n_i and m_i belong to the interval s_i, and $\sum_{i=1}^r n_i=n$, $\sum_{i=1}^r m_i=m$. It was shown that if F_1 and F_2 belong to a certain set of distributions, then for any positive ϵ, $P\{\phi(F_1, F_2)\geqq 1-\frac{1}{2}\epsilon^2\} > 1-(r^2+r-1)(1/\sqrt{n}+1/\sqrt{m})^4/\epsilon^4$ where $\phi(F_1, F_2)=\sum_{i=1}^r [(n_i/n)(m_i/m)]^{1/2}$.

24. *Estimation in the Poisson distribution when sample values of $c+1$ are sometimes erroneously reported as c*, by Professor A. C. Cohen, Jr., University of Georgia.

The method of maximum likelihood is employed to estimate the Poisson parameter when erroneous observation or reporting sometimes results in reporting as c, sample values of the random variable that are actually $c+1$. The proportion of erroneous observations is also estimated. Asymptotic variances and covariances of the estimates are obtained, and a numerical example is included.

25. *A note on the theory of Lagrange multipliers*, by Professor E. B. Shanks, Vanderbilt University.

In this note a theorem is proved which justifies the ordinary usage of Lagrange multipliers in determining stationary points of a function under a set of constraining relations. This note differs from other treatments in that the theorem states necessary and sufficient conditions, thus establishing the complete adequacy of Lagrange's method. Also, the theorem is proved without the use of differentials.

26. *On the inhomogeneous heat equation*, by Mr. D. G. Herr and Professor J. A. Nohel, Georgia Institute of Technology. (By title)

The equation (*) $u_{xx}-u_t=g(x, t)$, where g is a given function $(-\infty<x<\infty, t\geq 0)$ is studied under various assumptions regarding g. Results of Gevrey (J. Math. Pures Appl., 1914) and some extensions are obtained in a systematic fashion.

27. *A generalization of Hermite's differential equation*, by Professor Russell Cowan, University of Florida.

The differential equation contains two parameters so selected that if one is set equal to zero, the equation reduces to Hermite's differential equation. For certain integral values of the parameters polynomial solutions are obtained. These polynomial solutions are shown to be orthogonal and the integrated square is calculated. Several interesting relations between polynomial solutions are derived.

C. L. SEEBECK, JR., *Secretary*

THE APRIL MEETING OF THE SOUTHWESTERN SECTION

The annual meeting of the Southwestern Section of the Mathematical Association of America was held at the Air Force Missile Development Center, Holloman Air Force Base, New Mexico, on April 29–30, 1960. Professor R. B. Crouch, Chairman of the Section, presided at the afternoon session on April 29, and also at the morning session on April 30. There were 50 persons in attendance, including 35 members of the Association.

The following officers were elected: Chairman, Professor Harvey Cohn, University of Arizona; Vice-Chairman, Professor I. I. Kolodner, University of New Mexico; Secretary-Treasurer, Professor George Baldwin, New Mexico State University.

The following papers were presented:

1. *A remark on Pontrjagin duality*, by Professor George Baldwin, New Mexico State University.

Ellis (Trans. Amer. Math. Soc., vol. 83, 1956, pp. 301–312) showed "the complement of a Cartesian product of spaces, with the usual product topology, is the direct sum of their complements; and the complement of a direct sum of spaces, if its topology lies within a certain interval, is the Cartesian product of the complements." The upper bound of these topologies does not make the direct sum a topological group. The purpose of this remark is to replace the upper bound with a topology which makes the direct sum a topological group; specifically the Whitehead topology.

2. *Imbedding a group in a group with irreducible sets of generators*, by Mr. A. B. Gray, Jr., New Mexico State University, introduced by the Secretary.

It is shown that any group G can be imbedded in a group H such that G is a direct summand of H, H has an irreducible set of generators, and G/H is a direct sum of cyclic groups.

3. *On a representation of solutions of linear second order ordinary differential equations*, by Professor I. I. Kolodner, University of New Mexico.

Let $f \in C_p(I)$, $L = D^2 + f$. There exists a positive "amplitude" $u \in C_p^2(I)$ and a corresponding increasing "phase" $g \in C_p^3(I)$ such that every solution y of (*) $Lx = 0$ may be represented in the form $Y(t) = au(t)\cos(g(t) - \theta)$, a and θ constants. If (y_1, y_2) form a fundamental set of solutions of the differential equation (*) with Wronskian $\Delta > 0$, $\Delta =$ constant, then $u = \sqrt{(y_1 + y_2)}$, $g : g' = u^{-2}$ form such a pair; conversely to each pair (u, g) there exists a corresponding set (y_1, y_2). If $\Delta = 1$, the amplitudes form the class of positive solutions of $Lx = x^{-3}$. If $I \supset [0, \infty)$, $f = 1 + k$ and $\int_0^\infty |k(\tau)|\, d\tau < \infty$, there exists a unique "preferred amplitude" with property $\lim_{t \to \infty} u(t) = 1$. Applications are given.

4. *On the splitting of almost locally pure groups*, by Professor E. H. Walker, New Mexico State University.

Boyer and Walker have defined and made a preliminary investigation of almost locally pure groups. (Pacific J. Math., vol. 9, 1959, pp. 409–413.) One question left open is whether an almost locally pure group splits over its torsion subgroup. Using some results of Baer on extensions of Abelian groups, it is shown by using homological methods that there exists an almost locally pure group that does not split over its torsion subgroup.

5. *A further generalization of the Riesz theory of compact operators*, by Professor Seymour Goldberg, New Mexico State University, introduced by the Secretary.

It was shown by Graves (Trans. Amer. Math. Soc. vol 79, 1955, pp. 141–149) that if X and Y are Banach spaces, K a compact operator from X into Y and A a bounded linear operator from X onto Y, then a number of theorems which were proved by Riesz for the case where $X = Y$, $A = I$ could be generalized. In turn, a number of theorems which were proved by Graves are further generalized by replacing the operator A above by a closed linear operator defined on a domain dense in X and range all of Y.

6. *A general chain rule without components for derivatives of functions in vector spaces*, by Professor J. V. Lewis, University of New Mexico.

Ordinarily the chain rule for derivatives of functions in vector spaces must be stated in terms of partial derivatives and components. A general chain rule is developed without these limitations. Let f, g, h be such functions that $h(\mathbf{x})=g(f(\mathbf{x}))$ for every $\mathbf{x}$. $D_u h(\mathbf{x})=D_v g(y)D_u f(\mathbf{x})$ where $y=f(\mathbf{x})$ and v is a unit vector in direction $D_u f(\mathbf{x})$. Thus the derivative of h in direction u is the magnitude of the rate of change of f in that direction times the rate of change of g in the direction in which f is changing.

7. *Probability of ordered pairs in permutations*, by Mr. Milton Levy, White Sands Missile Range, Las Cruces, New Mexico.

If the elements $(1, \cdots, n)$ are permuted to $(a_1, \cdots, a_n)$, the resulting permutation contains an ordered pair if for some value of i, $a_i+1=a_{i+1}$. The permutation contains k pairs if there are k such values of i. Letting $c(n, k)$ be the number of n-permutations containing k pairs, then $c(n+1, k)=c(n, k-1)+(n-k)c(n, k)+(k+1)c(n, k+1)$ with boundary conditions: $c(n, k)=0$ if $k<0$ or $k\geq n$, $c(n, n-1)=1$. If each permutation is equally likely, then the probability of an (n, k)-permutation is $P(n, k)=c(n, k)/n!=[(n-k+1)/nk!]\sum_{j=2}^{n-k+1}[(-1)^j/j!]$.

8. *A remark on spectral theory*, by Professor J. B. Giever, New Mexico State University, introduced by the Secretary.

Let X be a complex Banach space, T a bounded linear operator on X to itself and $\sigma(T)$ its spectrum. Let A_T be the algebra of equivalence classes of complex functions analytic on a neighborhood of $\sigma(T)$ (two functions are equivalent if they agree on the intersection of their domains). This algebra has the additional useful structure of being the algebra of cross-sections of a sheaf F. Here F is the restriction to $\sigma(T)$ of the sheaf of Riemann surfaces over the complex plane.

9. *A theorem on commutators of symmetric groups*, by Mr. Don Cartlidge, New Mexico State University, introduced by the Secretary.

Let S be a set of cardinality A, where $A>\aleph_0$. Let $S(A, A^+)$ denote the permutation group on S and denote by $S(A, B)$ the set of permutations of $S(A, A^+)$ which move less than B elements of S. The author proved that if α is a fixed permutation of $S(A, A^+)$ with the property that, given any $\beta\in S(A, B)(\aleph_0<B<A^+)$ there exists a permutation γ of $S(A, A^+)$ such that $\beta=\alpha^{-1}\gamma^{-1}\alpha\gamma$, then the number of i-cycles of α is greater than or equal to B, for all $i=1, 2, \cdots, \aleph_0$.

10. *Some aspects of functional analysis and classical analysis*, by Professor A. E. Taylor, University of California, Los Angeles.

11. *Selecting beginning freshmen for calculus classes*, by Professor F. C. Gentry, University of New Mexico.

Many freshmen quite well qualified for a beginning course in analytic geometry and calculus on the basis of high school records and entrance tests fail to do satisfactory work. There seems to be a high correlation between first test grades and final grades in the course. Either a better criterion for selection should be found or students failing the first test should be given more preparation for the course.

12. *Changes in the University Mathematics Curriculum*, by Professor R. B. Crouch, New Mexico State University, Professor M. S. Hendrickson, University of New Mexico, and Professor E. D. Nering, University of Arizona.

DEONISIE TRIFAN, *Secretary*

THE APRIL MEETING OF THE TEXAS SECTION

The annual spring meeting of the Texas Section of the Mathematical Association of America was held on April 8–9, 1960, at San Antonio College, San Antonio, Texas. Professors W. T. Guy, Jr., University of Texas, and P. R. Culwell, San Antonio College,

presided. The total attendance was 153 including 105 members of the Association.

The officers for the next fiscal year are as follows: Chairman, Professor P. R. Culwell, San Antonio College; Vice-Chairman, Professor W. I. Layton, Stephen F. Austin State College; Secretary-Treasurer, Professor C. R. Sherer, Texas Christian University.

The following papers were given:

1. *The mapping function for a class of Riemann surfaces*, by Professor H. B. Curtis, Jr., University of Texas.

Because each member of a class of Riemann surfaces has been shown to be parabolic in type, a representation of the function which maps the plane conformally onto a surface of the class is obtained.

2. *The embedding of a lattice in a disjunctive lattice*, by Professor D. E. Edmondson, Southern Methodist University.

The theorem is proved that every lattice may be embedded in a disjunctive lattice in such a way that it is the homomorphic image of the disjunctive lattice in which it is embedded.

3. *An analysis of rounding errors in a solution of a symmetric tri-diagonal matrix*, by Mr. P. M. Blair, Rice Institute, introduced by the Secretary.

The partial differential equation $[a(x)u_x]_x = u_t$, arising from flow problems in one space dimension, may be solved by finite-difference methods. The tri-diagonal matrix obtained when an implicit difference equation is used has a solution which resists the growth of rounding errors.

4. *Use of a digital computer to build the multiplication table for the symmetric group on n elements*, by Professor H. A. Luther, Agricultural and Mechanical College of Texas.

By use of sequences whose terms are the consecutive numbers from 1 to n, n-factorial sequences are created. These sequences are then "multiplied" to give the usual table.

5. *Curves with bounded turning*, by Dr. Guy Johnson, Rice Institute.

A continuous curve γ with parametrization $z(t) = x(t) + iy(t)$, $a \leqq t \leqq b$, is said to have bounded turning if and only if $z'(t)$ is a function of bounded variation and $|z'(t)| \geqq \delta(\gamma) > 0$. The class of such functions is self-contained in the sense that a change of parameter $t = t(\tau)$, where $t'(\tau)$ is a function of bounded variation uniformly bounded from zero, yields again a curve of the same class. The arc length parametrization is always included. The name derives from the fact that $\theta(t) = \arg z'(t)$ is a function of bounded variation.

6. *Test for the rank of a matrix*, by Professor Louis Brand, University of Houston.

The $m \times n$ matrix

$$M = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}$$

is partitioned with $m = r + j$ and $n = r + k$. Then if the $r \times r$ submatrix P is nonsingular, M will be of rank r when and only when $RP^{-1}Q = S$. When this is the case, the system of m linear equations

$$\begin{pmatrix} P & Q \\ R & S \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

in n variables $x_1, \cdots, x_r, y_1, \cdots, y_k$ will be consistent when and only when $RP^{-1}a = b$. Here x, y, a, b, are respectively $r \times 1$, $k \times 1$, $r \times 1$, $j \times 1$ vectors. When consistent, the equations have the solution $x = P^{-1}a - P^{-1}Qy$, where y is an arbitrary column vector with k components.

7. *Concerning a generalization of convexity for functions of one variable*, by Mr. Larry Armijo, Rice Institute.

Convexity is defined with respect to the two parameter family of curves $A\phi_1(x) + B\phi_2(x)$, where $\phi_1(x)$ and $\phi_2(x)$ are continuous in the open interval (a, b) and are such that the two equations $y_1 = A\phi_1(x_1) + B\phi_2(x_1)$, $y_2 = A\phi_1(x_2) + B\phi_2(x_2)$, $x_1 \neq x_2$, are solvable uniquely for A and B. If $\phi_2(x)$ is

defined on (a, b), then ϕ_3 is convex if and only if it satisfies the inequality $D^0(\phi_i; x_i) \geqq 0$, $x_1 \leqq x_2 \leqq x_3$, where D^0 is the determinant $|\phi_i(x_i)|$, $i, j = 1, 2, 3$. Presuming that ϕ_i is a C^3-function, two other similar determinantal inequalities involving ϕ_i and its derivatives are obtained, and the relations among these three inequalities are examined.

8. *An interesting identity*, by Professor W. S. McCulley, Agricultural and Mechanical College of Texas.

The identity $(a^2+b^2)(c^2+d^2)=(ac+bd)^2+(ad-bc)^2$ and its generalizations are considered in relation to number theory, algebra, vector spaces and analysis. Uses of these identities by Fermat, Euler, Cauchy, Schwarz, Hurwitz, Gram, Schmidt, and others are mentioned, and their interpretations discussed. Proofs of various forms of the identity are given.

9. *Conformal invariants and quasiconformal mapping*, by Professor George Springer, University of Kansas.

10. *Some theorems on sequences*, by Professor R. K. Meany, Texas Christian University.

Three theorems on sequences are cited: 1) Mercer's Theorems; 2) A theorem of Erdös, Feller, and Pollard; 3) Some theorems of the author. In all these theorems the given sequence has the property that each term is influenced by or depends upon an average (in some sense) of the preceding terms, and from this property convergence is deduced.

11. *Backward continuation in time of the solution to the heat equation*, by Professor J. R. Cammon, Rice Institute, introduced by the Secretary.

When one wishes to predict a temperature distribution for preceding times from a presently known distribution by solutions of the heat equation, an improper boundary value problem must be solved. Such a problem was considered for the rectangle. By requiring the solution of the heat equation in the rectangle to be initially bounded, an a priori bound for the solution on compact subsets was derived, and an approximation by a finite linear combination of solutions was determined.

12. *A correction to a theorem on semi-metric topological spaces*, by Professor J. R. Boyd, Arlington State College.

A counterexample is presented which shows that the definition of the distance function as given in the argument for the statement "a Moore space is a semimetric topological space" (Pacific J. Math., vol. 6, 1956, p. 325) is not correct. A correction is given.

13. *Estimation of parameters for linear regression subject to certain restraints*, by Professor P. D. Minton and Mr. A. E. Crofts, Southern Methodist University.

Two cases of estimation of parameters of curves are considered: (1) Two straight lines constrained to meet at a known value of the independent variable; (b) A family of straight lines constrained to have a common slope or intercept; families of curves which can be transformed to a family of straight lines, *e.g.*, a family of exponentials with one common coefficient. Estimation formulas are given and examples are suggested.

14. *Systematic method for building the permutations of N numbers in a digital computer*, by Professor R. E. Kilmer, Agricultural and Mechanical College of Texas, introduced by the Secretary.

Permutations of N consecutive numbers, from 1 to N, may be built using N data storage locations in a digital computer by using N modulus arithmetic operations following a systematic rearrangement of the numbers. Permutations of N nonconsecutive numbers may be built using $3N$ data storage locations by using N consecutive numbers as index registers.

15. *An observed property of primitive roots of primes of the form* $4n-1$, by Professor Roger Osborn, University of Texas.

All primitive roots for odd primes less than 1000 were computed on the IBM 650. Several

properties of the primitive roots were observed and counted for each prime. It was observed that for most primes of the form $4n-1$ more than half of the primitive roots exceed $\frac{1}{2}p$. A parallel with the occurrence of quadratic residues can be observed.

16. *The mathematics program in the small college*, by Professor W. I. Layton, Stephen F. Austin State College.

A discussion of course offerings in mathematics and requirements for majoring and minoring in mathematics at Stephen F. Austin State College are given.

17. *The justification for a new course*, by Professor R. S. Underwood, Texas Technological College.

Assuming that a practical plane analytic geometry for n variables can be developed, the question here discussed is whether or not this theory justifies a separate undergraduate college course. In considering it we should note that the two surfaces used chiefly in the preservation and diffusion of knowledge are the printed page and the blackboard. The high usefulness of the geometry of space does not automatically warrant the exclusion of a parallel development on the plane, with its own peculiar and obvious advantages. In line with this premise, a textbook has been prepared for a course now being taught.

18. *SMSG twelfth-grade course*, by Professor L. K. Durst, Rice Institute.

19. *The national high school mathematics contest*, by Professor W. H. Fagerstrom, Pan American College.

The Committee on the National High School Contest is an outgrowth of the Contest Committee of the Metropolitan New York Section, which was organized on October 30, 1949. Dr. W. H. Fagerstrom of the City College of New York was elected Chairman. The first contest was held May, 1950, with less than 5,000 students from approximately 200 schools. This contest has grown steadily since its organization. At the Seattle meeting of the Association, on the suggestion of the Contest Chairman, the contest became international under the joint sponsorship of the Association and the Society of Actuaries. The 1960 contest has over 150,000 students from about 5,000 schools in the United States and Canada.

20. *What, besides mathematics, should the mathematics teacher be teaching?* by Professor E. R. Heineman, Texas Technological College.

The mathematics teacher, not only in the secondary school but also in the college, should (1) teach the student how to learn by himself, (2) encourage the student to reason and think logically, and (3) insist upon precise language in the description of and the solution of mathematical problems.

21. *Research on ordinary linear differential equation of second order*, by Visiting Professor Ping-Chang Van, Texas Technological College, introduced by the Secretary.

Part I, FACTOR THEOREM AND FORMULAS: A given equation can be factorized and formulated, if one of the two complementary functions is known. Part II, FORMULA AND PROOF FOR THE METHOD BY VARIATION OF PARAMETERS. Solving the two formulas for the particular integral, one of the two unknowns is the required particular integral and the other is the proof. Part III, PROOFS FOR CO-I.F.E. THEOREM: There exists another equation corresponding to the given one, each complementary function of the one being the integrating factor of the other; three proofs. Part IV, CONDITION THEOREM CONCERNING (1) $f^2(x)+F^2(x)=c^2$ and (2) $f(x)F(x)=c^2$, where $f(x)$ and $F(x)$ are the two complementary functions of the given equation and c a constant. The solution can be given by formula.

C. R. SHERER, *Secretary*

THE MAY MEETING OF THE ILLINOIS SECTION

The thirty-ninth annual meeting of the Illinois Section of the Mathematical Association of America was held at Illinois Wesleyan University, Bloomington, Illinois, on

May 13–14, 1960. Professor D. E. Myers, Chairman of the Section, presided at all sessions. There were 80 persons in attendance, including 58 members of the Association.

Reports were given by the Committee on the Strengthening of the Teaching of Mathematics, the Committee on Contests and Awards, and the Committee on the Role of the Section. The Section approved the recommendation to conduct the High School Lecture Program for another year and instructed the Chairman to appoint a committee to do this. The Illinois Section will pay the expenses of this program. The following officers were elected to serve for the coming year: Chairman, Professor Douglas Daly, Illinois Wesleyan University; Vice-Chairman, Professor Toivo Rine, Illinois State Normal University; Secretary-Treasurer, Professor Wayne McGaughey, Bradley University.

Following a brief welcome by Dean John Smith of Illinois Wesleyan University, the following program was presented:

1. *Information theory*, by Professor David Blackwell, University of California, Berkeley (by invitation)

An exposition of some of the work of Shannon, Feinstein, and others.

2. *Probabilistic methods in number theory*, by Professor W. J. LeVeque, University of Michigan (by invitation)

From among the many applications of probability theory to number theory, the following three were discussed and compared: a) The heuristic principle that the property (or "event") of primality of a positive integer is nearly independent of a variety of other interesting properties which the integer may possess. b) The application of the central limit theorem to additive number-theoretic functions, as initiated by P. Erdös and M. Kac. c) The extension of the various limit theorems of probability theory from the case of independent variables to certain dependent sequences, all elements of which are defined in terms of a single random variable uniformly distributed on $[0, 1]$. The results are metric theorems describing the "usual" behavior of decimal digits, continued fraction denominators, etc.

3. *Relativity and space travel*, by Professor M. C. Moore, Bradley University (by invitation)

The nature of matter does not allow it to move faster than light. However, future man can still travel over enormous distances, for relativity predicts that he will live more slowly at high speeds. With an apparent acceleration never exceeding that of gravity, he could cross our galaxy and return in a time which for him would be only 45 years, although 200,000 years would have elapsed at home. In half the 45 years, if he never decelerated, he could travel beyond the reach of our largest telescopes. Slides showing diagrams, equations, and astronomical scenes illustrated the talk.

4. *Elementary statistics, an experimental course*, by Professor D. R. Bey, Illinois State Normal University.

Over a period of five years, the author has been experimenting with various topics and sequences of topics suitable for a beginning course in statistics open to students without calculus. Many of the topics included in this course are discussed. Although the direction of the course is not entirely determined, its emphasis is on proofs and mathematical structure.

5. *Elementary Russian mathematics*, by Professor G. H. Miller, Western Illinois University.

An analysis of the Russian mathematics program corresponding to our elementary program indicates that their students cover material which is completed one to two years later in our schools. A basic introduction to such concepts as approximation, qualities, inequalities, and the fundamental laws (commutative, distributive, associative) are considered in the fifth and sixth grades. The official texts used in Russia have a greater emphasis on statement problems and oral computation. Their program is oriented so that the students are provided with a background for higher mathematics.

A. W. McGAUGHEY, *Secretary*

THE MAY MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The annual Spring Meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held at the University of Virginia, Charlottesville, on Saturday, May 7, 1960. Eighty persons were present, including 70 members of the Association. Professor E. E. Floyd, Chairman of the Section, presided. Professor C. H. Frick, Chairman of the Nominating Committee, placed in nomination the following slate of officers for the coming year: Chairman, Professor D. B. Lloyd, District of Columbia Teachers College; Vice-Chairmen, Professor Mary E. Hamstrom, Goucher College, and Professor S. S. Saslaw, Naval Academy; Secretary, Professor Herta T. Freitag, Hollins College; Treasurer, Professor S. B. Jackson, University of Maryland. They were unanimously elected.

The following papers were presented:

1. *Graphical method for the determination of ionic sites with preferential fixation*, by Dr. Claude Marmasse, Hollins College.

A graphical method for the evaluation of a sum of inverses is used. It concerns the physical theory of elementary processes in the case of the ionization of a polyelectrolyte macromolecule.

2. *On generalized K-nions loops and algebras*, by Professor Volodymyr Bohun-Chudyniv, Morgan State College.

The aims of this paper were: (1) to construct methods of determining the orders of all possible types of the loops, groups, and algebras, which differ from one another at least in the multiplication table; and each multiplication table can be represented by triplets, as in case of K-nions type; (2) to determine the all nonisomorphical types of generalized K-nion algebras and groups of the 10th order, and construct examples for 14th, 20th and other orders; (3) to point out an application of the generalized K-nions algebras for the determining of all types of diagonal Latin squares of the corresponding orders.

3. *Fourier series and eigenfunction expansions associated with a non-self-adjoint differential equation*, by Professor L. I. Mishoe, Morgan State College.

Fourier series may be considered as an eigenfunction expansion associated with the self-adjoint equation: $u''+\lambda u=0$. While Fourier series has been studied with great interest by several mathematicians, apparently no studies have been made connecting expansions associated with self-adjoint equations with expansions for non-self-adjoint equations.

In this paper the non-self-adjoint equation (*) $u''+q(x)+\lambda[p(x)u-u']=0$, where $u(0)=u(1)=0$, is considered. It is shown that if $u_n(x)$ are eigenfunctions of (*) and a_n eigencoefficients determined in the usual manner, then: $\sum_{-\infty}^{\infty} a_n u_n(x)$ is convergent, divergent, or uniformly convergent with the Fourier series $\sum_{-\infty}^{\infty} \exp(2n\pi ix)\int_0^1 h(\xi)\exp(-2n\pi i\xi)d\xi$, where $h(x)=f(x)\exp(g(x))$ for $f(x)$ of class $L^2(0, 1)$ and $g(x)$ is a function of p in (*).

4. *Dissection of the nonacute triangle and the square into acute triangles*, by Mr. P. J. Federico, Examiner-in-Chief, Patent Office, Washington, D. C.

5. *Fourier transforms*, by Professor E. J. McShane, University of Virginia.

For many applications, the theory of the Fourier transform should be based on convergence in the mean; for undergraduates only the Riemann integral is available. In this talk it was shown that if f is square-integrable and has a Fourier transform g, then $|f|^2$ and $|g|^2$ have the same (Riemann) integral over the real axis; and if in addition f has only finitely many discontinuities in any finite interval, the conjugate Fourier transform of the g is f.

6. *Geometrical theorems for abscissas and weights of Gauss type*, by Dr. P. J. Davis, Chief, Numerical Analysis Section, National Bureau of Standards, Washington, D. C.

It is commonplace to apply rules of numerical analysis to computation. It is something of a

novelty to see computation applied to rules of numerical analysis. From tables of Gauss, Laguerre, Hermite, and Lobatto quadrature formulas which have recently been computed with great accuracy, and using the "method of inspection," P. Rabinowitz and the lecturer have formulated and proved a number of asymptotic results relating the abscissas to the weights. Other conjectures, including one by G. Szegö, seem plausible numerically, but proofs are still to be supplied.

7. *Nets and calculus*, by Professor B. J. Pettis, University of North Carolina. (By invitation)

D. B. LLOYD, *Secretary*

THE MAY MEETING OF THE ROCKY MOUNTAIN SECTION

The forty-third annual meeting of the Rocky Mountain Section of the Mathematical Association of America was held at the United States Air Force Academy, Colorado Springs, Colorado, May 6 and 7, 1960. On Saturday the Section enjoyed a joint meeting and lecture with the Rocky Mountain Section of the Society for Industrial and Applied Mathematics. The meeting was divided into several sessions with Professors J. W. Ault, R. M. Elrick, J. W. Querry, J. S. Leech, and A. W. Recht presiding. There were 132 persons registered for the meeting, including 102 members of the Association.

Officers elected at the meeting for 1960–1961 were: Chairman, Professor L. W. Rutland, University of Colorado; Vice-Chairman, Professor L. C. Barrett, South Dakota School of Mines and Technology; Secretary-Treasurer, Professor Leota Hayward, Colorado State University; and Director of High School Mathematics Contest, Professor D. C. B. Marsh, Colorado School of Mines.

The following papers were presented:

1. *Cauchy's theory of characteristics*, by Professor R. W. McKelvey, University of Colorado.

This was a discussion of the recent studies of A. Plis and T. Wazewski on the domain of existence of solutions of the first order nonlinear partial differential equation $F(x, u, u_x)=0$, $x=(x_0, \cdots, x_n)$. Wazewski's derivation [Bull. Acad. Polon. Sci. Cl. III No. 4, 1956, pp. 131–135] of Plis' results [same source, pp. 125–129] is in the context of the classical Cauchy theory of characteristics. The method can be applied, by analogy, to the quasi-linear equation

$$\sum a_i(x, u)u_{x_i}, -b(x, u) = 0,$$

and is simple enough to find a place in elementary textbooks.

2. *Variation of parameters by vector methods*, by Professors L. C. Barrett and C. A. Grimm, South Dakota School of Mines and Technology, presented by Professor Grimm.

For the equation, $a_0y'''+a_1y''+a_2y'+a_3y=f(x)$, where the a's and $f(x)$ are functions of x continuous on a closed interval over which $a_0\neq 0$, whose reduced equation has the independent solutions y_1, y_2, y_3, we assume the particular solution $y=u_1y_1+u_2y_2+u_2y_3=\vec{U}\cdot\vec{Y}$, $\vec{U}$ to be determined by substitution. From this dot product representation it follows immediately that $y=\int_{x_0}^{x}[\vec{Y}(x)\cdot\vec{w}(t)f(t)][a_0(t)W(t)^{-1}]dt$, $\vec{w}=\vec{Y}\times\vec{Y}'$, $W=\vec{w}\cdot\vec{Y}''$, the Wronskian. It was shown how to generalize the results to other orders of equations.

3. *Existence and stability of periodic solutions of weakly nonlinear differential equations*, by Dr. H. R. Bailey, The Ohio Oil Company, Denver Research Center.

Existence and stability theorems for periodic solutions of weakly nonlinear differential systems have been given recently in a number of papers using a convergent method of successive approximations. This method was originally considered by Lamberto Cesari in 1940 for linear systems.

In the present paper the existence and stability theorems are specialized to the case of a weakly nonlinear differential equation and then applied to a nonlinear Mathieu equation. A summary of the corresponding results in more general situations is given.

4. *On solutions of second order differential equations by changing variables*, by Professor F. M. Stein and Mr. R. D. Finley, Colorado State University, presented by Mr. Finley.

In this paper the authors consider the various methods of changing the dependent and independent variables in the equation $y''+P(x)y'+Q(x)y=0$, where it is assumed that the solutions are known to exist, to reduce it to some classical form or to an equation which may be solved by standard means.

5. *Particular solutions for systems of:* (1) *nonhomogeneous, linear, ordinary differential equations,* (2) *nonhomogeneous, linear, ordinary difference equations*, by Professors L. C. Barrett and R. A. Jacobson, South Dakota School of Mines and Technology, presented by Professor Barrett.

In this paper Lagrange's identity and the bilinear concomitant, so familiar in ordinary differential equation theory, are generalized and then applied, as an alternative to variations of parameters, in solving systems of nonhomogeneous linear ordinary differential equations. A parallel treatment of difference equations is also given.

6. *An integral transform*, by Professor F. M. Hudson, Western State College.

If it is possible to write a differential equation in the form (1) $g(gF')'+gF'+F=0$, where g represents any function of x, then the solution of the equation can usually be found by use of the integral transform $T[F(x)]=\int_0^\infty K(m,x)F(x)dx$. The kernel $K(m,x)$ will depend on g and will be given by the equation $K(m,x)=g^{-1}\exp[\int-(m/g)dx]$. This transform will have as its basic differentiation property $T[gF']=mT[F]$. If an equation is written in the form (1), then the most convenient transform will be suggested by the function g. The Laplace and Mellin Transforms are special cases of the transform $T[F(x)]$.

7. *School Mathematics Study Group experimentation in Colorado*, by Professors W. E. Briggs and L. W. Rutland, Jr., University of Colorado, presented by Professor Rutland.

The objectives and accomplishments of the School Mathematics Study Group and of the Colorado Junior High Center and Colorado Geometry Center were reviewed. An outline was given of the SMSG texts for grades seven through twelve.

8. *The power series coefficients of certain L-functions*, by Professor W. E. Briggs, University of Colorado, and Professor R. G. Bushman, Oregon State College, presented by Professor Briggs.

An example is given to indicate a general method for determining the power series coefficients of functions defined by Dirichlet series. If χ is a principal character modulo k and $h=\phi(k)/k$, then $L(s)=\sum_{n=1}^{\infty}\chi(n)n^{-s}$ can be written as $h/(s-1)+\sum_{r=0}^{\infty}(-1)^rV_r(s-1)^r/r!$. Let $L_1(s)=L(s)-hs/(s-1)$. Using an integral representation of $L(s)$, the authors derive the Theorem: *If* $u<0$, *then* $\sum_{n\leq x}n^u\chi(n)\log^r n=h\int_1^x t^u\log^r t\,dt+(-1)^rL_1^{(r)}(-u)+0(1)$. By setting $u=-1$ and letting $x\to\infty$, it follows that $V_0=\lim_{x\to\infty}[\sum_{n\leq x}n^{-1}\chi(n)-h\log x]+h$ and

$$V_r=\lim_{x\to\infty}\left[\sum_{n\leq x}n^{-1}\chi(n)\log^r n-\{h/(r+1)\}\log^{r+1}x\right],\qquad r=1,2,\cdots.$$

9. *New teacher education program in mathematics at Colorado State College*, by Professor D. O. Patterson, Colorado State College.

The program in mathematics at Colorado State College is to become, beginning in September 1960, more extensive in its offering and follow fairly closely the recommendations of the National Commission on Mathematics. New courses for secondary school teachers include such titles as "set theory," "modern algebra," "probability theory," "analysis," and "statistics." For the elementary school teachers new courses in mathematics are "arithmetic for elementary teachers" and "foundations of arithmetic."

10. *Astronautics program at USAF Academy*, by Colonel R. C. Gibson, USAF Academy.

One of the principal values of the astronautics program at the Air Force Academy is pedagogic, in that mathematics, physics and chemistry are brought together in a challenging and timely way during the two senior semesters.

The astronautics courses are using most, if not all of the mathematics taught in the regular cadet mathematics sequence. Both the mathematics department and the astronautics department are finding it extremely challenging to design the courses to maximize the cadet's ability to use and appreciate the magnificence of mathematics.

11. *Are mathematicians playing fair with Uncle Sam?* by Professor A. W. Recht, University of Denver.

Recent Russian advances in science forced a hysterical Uncle Sam into a crash program to support so-called "modern mathematics." NSF millions have been spent, and perhaps squandered, to support mathematical study groups and institutes to indoctrinate high school teachers. College professors have gone on a binge of riding their mathematical hobbies at government expense. Payola in mathematics is not on the grand scale of TV payola and rigged programs, but has had a profound effect on the morale of mathematics education. Have the mathematical touts had Uncle Sam gamble on the wrong horses? Have the mathematicians a real solution or a racket?

12. *Applications of mathematics in technology*, by Professor N. W. McLachlan, Visiting Professor of Applied Mathematics, University of Colorado (Invited address)

13. *Anti-associative systems*, by Professor C. H. Cunkle and Mr. D. R. Rogers, Utah State University, presented by Mr. Rogers.

A binary operation defined on a set S is anti-associative provided that $a(bc)\neq(ab)c$ for each a, b, c in S. Examples of anti-associative systems of n elements ($n>1$) with one binary operation were constructed, and these were characterized for sets of 2 and 3 elements. A generalization of associativity was defined for binary operations mapping a finite set onto itself. These operations fell into classes which formed the elements of a permutation group whose identity element was the class of associative operations.

14. *A new nomographic treatment of quartic equations*, by Professor C. R. Wylie, Jr. and Mr. Elbert Johnson, University of Utah, presented by Professor Wylie.

The equation $\phi_1(t)+a\phi_2(t)+b\phi_3(t)+c=0$ can be written as the pair of equations $\phi_1(t)+a\phi_2(t)-p=0$, $b\phi_3(t)+c+p=0$, for each of which a nomogram can be constructed, provided that in the second equation $c+p$ be treated as one of the variables. The contributions of this paper are (1) a description of a mechanical device to facilitate the simultaneous use of these nomograms in the trial and error case when a, b, and c are given and t is required, and (2) the application of this procedure to the solution of the quartic equations $Ut^4+t^3+Vt^2\pm t+W=0$ which are obtained from the general quartic equation $x^4+a_1x^3+a_2x^2+a_3x+a_4=0$ via the substitutions $x=(\pm a_3/a_1)^{1/2}t$, the plus or minus sign being chosen according as $a_1a_3>0$ or $a_1a_3<0$.

15. *Probability in differential equations*, by Captain R. L. Eisenman, USAF Academy.

The solution $y=a-b\cos(Ct+d)$, where a, b, d are constants and C is a random variable, is eventually distributed as $F(k)=\cos(a-k)/b$ regardless of the distribution of C. The problem was motivated by perturbation of a satellite in circular orbit.

16. *An elementary method of determining initial conditions for missile trajectories*, by Professor C. H. Cunkle, Utah State University.

A missile trajectory can be predicted by solving a set of simultaneous differential equations. Such solutions can be very costly, and, in the process of designing a missile, methods are sought for reducing the number of solutions required. By assuming that the range is a quadratic function and using three-point interpolation, the proper initial conditions for a desired range are approximated in a minimum number of trials. The process involves only elementary analytic geometry.

17. *A matrix application of Newton's identities*, by Professor D. W. Robinson, Brigham Young University.

A novel proof, which is based upon Newton's identities, is given of the following theorem. *Let A be an n-by-n matrix over a field of characteristic zero or prime $p>n$. Then A is nilpotent if and only if trace $A^k=0$, $k=1, \cdots, n$.* Other applications are also suggested.

18. *An application of Markov processes in inventory theory*, by Professor P. W. Zehna, Colorado State College.

In recent investigations in the area of inventory depletion, it was possible to define a sequential model for issuing items from a stockpile. By considering the ages of the items to be random variables, a stochastic process was determined for each of two issuing schemes that were of interest. Further examination revealed imbedded Markov processes in each scheme. It was then possible to find unique stationary absolute probability distributions for each scheme and compare them on the basis of the statistical equilibrium afforded by the stationary distributions.

19. *Reflections of a mathematician*, by Professor L. J. Mordell, Visiting Professor of Mathematics, University of Colorado. (Invited address—SIAM)

The topics discussed were selected from the following: What is mathematics, and what are the difficulties in its study? How are mathematicians made, and how do they work? How do problems arise, and how are they solved? What help is given by the electronic computers? What part is played by memory and luck, and what kind of mistakes and errors do mathematicians make? Finally there are the aesthetic and international aspects of mathematics.

The following papers were presented by title:

20. *Limits of iterated discontinuous functions*, by Professor Emeritus A. J. Kempner, University of Colorado.

21. *On Whitworth's Exercise 667*, by Lt. D. R. Barr, USAF Academy.

At least four solutions of problems equivalent to Exercise 667 in Whitworth's *DCC Exercises in Choice and Chance* have appeared in the literature since the publication of the latter in 1897 (see references in J. O. Irwin, J. Roy Statist. Soc., vol. 118, pp. 393–396). A new solution, using only basic definitions and the formula for the simultaneous occurrence of exactly m among N events, is presented.

22. *Green functions for systems of differential equations*, by Professors L. C. Barrett and R. A. Jacobson, South Dakota School of Mines and Technology.

The present note is concerned with extending the familiar concept and usage of Green's function to nth order systems of ordinary linear differential equations with two-point boundary conditions.

23. *Methods for calculating principal idempotents*, by Mr. J. C. Higgins, Brigham Young University.

24. *On a generalization of Pythagorean theorem*, by Professor Aboulghassem Zirankzade, University of Colorado.

The generalized Pythagorean theorem, in Euclidean n-space, is well known and a simple proof, using methods of vector analysis, exists. A more elementary proof, using only plane and solid Euclidean geometry, is given for the case $n=3$ or $n=4$. It is also shown how this method could be modified to prove the generalized theorem for the case $n>4$. Because of the existence of this simple proof, the generalized theorem becomes a suitable topic to be offered in secondary school.

F. M. CARPENTER, *Secretary*

THE MAY MEETING OF THE WISCONSIN SECTION

The twenty-eighth annual meeting of the Wisconsin Section of the Mathematical Association of America was held at Mount Mary College, Milwaukee, Wisconsin, on May 7, 1960. Professor C. B. Hanneken, Chairman of the Section, presided. This meeting was held jointly with the May meeting of the Wisconsin Mathematics Council and there were 129 present, including 52 members of the Association and 62 members of the Wisconsin Mathematics Council.

At the business meeting, the following officers were elected for the coming year: Chairman, Professor Henry Van Engen, University of Wisconsin; Vice-Chairman, Professor Earl Swokowski, Marquette University; Secretary-Treasurer, Dr. E. F. Wilde, Beloit College.

The following papers were presented:

Morning Session for members of the Association

1. *A combinatorial problem on integers*, by Professor Frank Wagner, Marquette University.

Let $0 \leqq \alpha \leqq 9$ and $x > 1$ be integers. Let $p(\alpha, x)$ be the number of occurrences of the digit α in the list of integers from 1 up to but not including x. If $x = \sum_{i=0}^{n} \beta_i 10^i$, $0 \leqq \beta_i \leqq 9$, let $B_k = \sum_{i=k}^{n} \beta_i 10^i$. Then $B_n = \beta_n 10^n$, $B_0 = x$. We obtain the relation

$$p(\alpha, B_k) - p(\alpha, B_{k+1}) = p(\alpha, \beta_k 10^k) + \beta_k 10^k \sum_{j=k+1}^{n} \delta_{\alpha\beta_j},$$

where δ is the Kronecker δ. From this we get

$$p(\alpha, x) = \sum_{1} j\beta_j 10^j + \sum_{2} 10^j + \sum_{3} \sum_{i=0}^{j-1} \beta_i 10^i,$$

where the summation ranges in $\sum_1$, $\sum_2$, $\sum_3$, are $0 \leqq j \leqq n$; $\beta_i > \alpha$, $0 \leqq j \leqq n$; $\beta_i = \alpha$, $1 \leqq j \leqq n$; respectively. If $\alpha = 0$, from this last we must subtract $(10^{n+2} - 9n - 19)/81$.

2. *Analytic mappings of a finite area*, by Mr. Daniel Waterman, University of Wisconsin, Milwaukee.

The condition that an analytic function map a domain in the plane onto a region of finite area on its Riemann surface, was the problem under consideration. The Tauberian nature of this condition was discussed and the domains considered were subsets of the circle, part of whose boundary is an arc of the circle and regions in the half-plane of convergence of a Laplace transform with intervals on the abscissa of convergence contained in the boundary.

3. *Some common problems in college mathematics programs*, by Dr. Lawrence Wahlstrom, Wisconsin State College, Eau Claire.

The author listed ten questions for discussion by a proposed Wisconsin Committee for Mathematics in the State Colleges. By meeting regularly for a sufficient length of time, the committee could come to grips with such problems as (1) desirable changes in the sequence of mathematics courses for prospective mathematics teachers; (2) a common proficiency examination; (3) courses and course-content requirements; (4) developing effective liaison with public school teachers; (5) the graduate program in mathematics; (6) pooling staff resources for an effective research program in the teaching of mathematics. This "Wisconsin-idea" would be advisory and not obligatory in actual operation.

4. *Geometry and group theory*, by Dr. G. O. Losey, University of Wisconsin.

The purpose of this exposition was two-fold: (1) to illustrate how the fundamental ideas of group theory may be introduced by the consideration of the group of allowable transformations of a geometry; (2) to indicate the nature of recent studies on the ways that abstract group theorists are using the language and techniques of geometry to learn more about groups. The principal example for part (1) was the group of collineations of a projective plane. In part (2) the work of N. S. Mendelsohn, G. Bachman, and D. G. Higman and J. B. McLaughlin was discussed.

Morning Session—Wisconsin Mathematics Council

5. *The ninth-grade course of the School Mathematics Study Group*, by Mr. Henry Swain, New Trier Township High School, Winnetka, Illinois.

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about a central theme, the structure of the real number system. All the usual manipulations of algebra are found to depend on a few basic properties of numbers. Some proving is done to give some simple experience with proofs, not to make the course completely rigorous. The language of sets, the use of the number line, and absolute values, and the treatment of inequalities all contribute to a course which interests the students, is teachable and is mathematically sound.

Afternoon Session—joint session of the Wisconsin Section M. A. A. and the Wisconsin Mathematics Council.

6. *Geometric algebra*, by Professor Ernst Snapper, Indiana University.

Geometric algebra is a term coined by Artin who wrote a book by this title. This term conveys the fact that the modern algebraist replaces as much as possible concrete algebraic objects like polynomials, matrices, etc., by the abstract geometric concepts which they represent. As an example of this geometric attitude it was discussed how the theory of quadratic forms should be developed in terms of "metric vector spaces." Witt's theorem was discussed as an example of a theorem which could not even be discovered until after the geometric attitude had been adopted.

E. F. WILDE, *Secretary*

CALENDAR OF FUTURE MEETINGS

Forty-fourth Annual Meeting, Willard Hotel, Washington, D. C., January 25–27, 1961.

Forty-second Summer Meeting, Oklahoma State University, Stillwater, Oklahoma, August 28–31, 1961.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, West Virginia University, Morgantown, May 6, 1961.

ILLINOIS, University of Illinois, Urbana, May 12–13, 1961.

INDIANA

IOWA, State University of Iowa, Iowa City, October 10–11, 1960.

KANSAS

KENTUCKY, Western Kentucky State College, Bowling Green, Spring, 1961.

LOUISIANA-MISSISSIPPI, Buena Vista Hotel, Biloxi, Mississippi, February 17–18, 1961.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, University of Maryland, College Park, December 3, 1960.

METROPOLITAN NEW YORK

MICHIGAN, Wayne State University, Detroit, March 25, 1961.

MINNESOTA, University of North Dakota, Grand Forks, Fall, 1960.

MISSOURI, University of Missouri, Columbia, Spring, 1961.

NEBRASKA, University of Nebraska, Lincoln, April 15, 1961.

NEW JERSEY, Rutgers, The State University, New Brunswick, November 5, 1960.

NORTHEASTERN, Wesleyan University, Middletown, Connecticut, November 26, 1960.

NORTHERN CALIFORNIA, San Jose State College, January 14, 1961.

OHIO, Ohio Wesleyan University, Delaware, May 6, 1961.

OKLAHOMA, Oklahoma City University, October 21–22, 1960.

PACIFIC NORTHWEST, University of Washington, Seattle, June 17, 1961.

PHILADELPHIA, Swarthmore College, Swarthmore, Pennsylvania, November 26, 1960.

ROCKY MOUNTAIN, University of Colorado, Boulder, May, 1961.

SOUTHEASTERN, Wofford College, Spartanburg, South Carolina, April 7–8, 1961.

SOUTHERN CALIFORNIA, University of California, Santa Barbara, March 11, 1961.

SOUTHWESTERN, University of Arizona, Tucson, April, 1961.

TEXAS, Stephen F. Austin State College, Nacogdoches, April 14–15, 1961.

UPPER NEW YORK STATE, Harpur College, Endicott, April 29, 1961.

WISCONSIN, University of Wisconsin, Madison, May, 1961.

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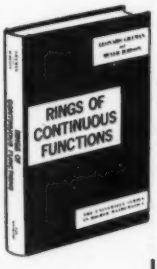

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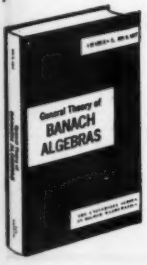

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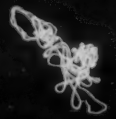

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VOLUME 67 NUMBER 7

PART II

Elementary Point Set Topology

R. H. BING

Number 8
of the
HERBERT ELLSWORTH SLAUGHT
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AUGUST-SEPTEMBER 1960

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ELEMENTARY POINT SET TOPOLOGY

By

R. H. BING

Professor of Mathematics

The University of Wisconsin

The Eighth

HERBERT ELLSWORTH SLAUGHT

MEMORIAL PAPER

Published as a supplement to the AMERICAN MATHEMATICAL MONTHLY

Volume 67 Number 7

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1. INTRODUCTION

Topology as an offshoot of geometry. In many respects, topology may be considered an offshoot of geometry. It is a study of properties that are somewhat geometric. Just as it would be difficult to define geometry to a person unacquainted with such terms as points, lines, triangles, and planes, it is difficult to give an understandable definition of topology without using some of the terminology of topology. We delay giving a definition of topology until later.

A person may approach the study of topology in somewhat the same spirit that a student begins the study of synthetic Euclidean plane geometry. The student brings into his geometry class a rich background of information about the Euclidean plane. There is a listing of certain properties of the plane. This list is called a set of *postulates* or *axioms*. An effort is made to make the list so complete that the theorems about the plane can be proved on the basis of the postulates alone without making any recourse whatever to our intuitive knowledge about the plane. We strive for efficiency in our list of postulates by eliminating any postulate that can easily be proved from the remaining postulates.

The study of topology frequently starts with a set of postulates or axioms. In a later section we list some axioms that can start us on a study of topology. In plane geometry we use the plane as a model to suggest the postulates. The plane also acts as a model to suggest our beginning axioms for topology. However, a line also makes a good model for these axioms. There are other models as suggested in a later section *Some examples of topological spaces*.

In geometry we try to give explicit definitions of the terms we use. This is in addition to the descriptive discussion accompanied by examples that convey additional information. Explicit *definitions* describe the thing being defined in terms of things whose meaning is already assumed to be understood. This is different from the definitions as given in dictionaries where a "mesh-type" of definition is given and there is no clear understanding as to which term is defined first, but enough synonyms are included so that a person who knows the meaning of many other words is likely to have enough information to decipher the meaning of the word under consideration. In mathematics, definitions are given in a "tree-like" fashion where a term is defined by using previously ac-

cepted terms. We try to avoid the "circular" lines of definitions found in dictionaries.

The basic terms that we start with are not defined in terms of previously defined terms. These are our *undefined terms*. If there is to be a beginning, there must be some term used first. Perhaps a more compelling reason for having undefined terms is that although we may have an intuitive feeling as to the meaning of a basic term under consideration, any meaningful formulation of its meaning involves more complicated terms. For example, the fuzzy definition "A point is that which has position but no dimension" may be criticized on the basis that "position" and "dimension" were used. A similar criticism may be made of the definition "A point is the intersection of two nonparallel lines." In topology we use "point" as an undefined term. In the last section of this article we mention another advantage of undefined terms in topology—namely, if a term is left undefined, any meaning can be assigned to it and as long as the axioms are still satisfied, the theorems we have proved on the basis of the axioms hold true with this meaning. This results in richer interpretations of our theorems.

The set of all points under consideration is called *space*. If we are studying plane geometry, we take the Euclidean plane E^2 to be space. In studying solid geometry, we take Euclidean 3-space E^3 to be space. In topology it is customary to take a set of axioms that will hold in many different spaces and the theorems we prove will hold true in each of the spaces. We suppose that the reader started this chapter with a knowledge of the Euclidean line E^1, the Euclidean plane E^2, and Euclidean 3-space E^3. One section of this article is devoted to a listing of various other spaces.

In geometry we prove theorems on the basis of some axioms, definitions, and undefined terms. We do the same thing in topology—although we use different axioms, undefined terms, and definitions. This schematic diagram shows the axiomatic approach.

$$\left.\begin{array}{l}\text{Undefined terms}\\ \quad\text{(point, neighborhood, . . .)}\\ \text{Definitions}\\ \quad\text{(limit point, closed, topologically equivalent, . . .)}\\ \text{Axioms}\end{array}\right\}\Rightarrow\text{Theorems}$$

Before considering a set of axioms for topology, we shall indulge in an intuitive treatment of several aspects of topology. Our treatment contains some rubber-sheet geometry that may be of interest in spite of the fact that it does not add much to our knowledge of topology. We hope that by mentioning the axiomatic approach in this introduction and by returning to it after a brief sojourn to enrich our topological background in which some topological oddities are mentioned, we leave the impression that the true spirit of topology lies in proving theorems rather than in the oddities.

To point out things that are studied in topology but not in geometry we consider the simple closed curves shown in Figure 1. Geometrically they are

FIG. 1

different. However they are topologically equivalent. In a course in topology we study properties common to such figures. We list three topological properties of a simple closed curve.

1. A simple closed curve is connected. (We give a definition later of "connected" but our intuitive notion of "being in one piece" may suffice here.)
2. If a point is removed from a simple closed curve, the remaining part is connected. (Although a person could not physically remove a point from an object, we can do the operation mentally.)
3. If two points are removed from a simple closed curve, the remaining part is not connected but is in two pieces. (These pieces are called "components" of the remainder.)

Exercise 1. If three points are removed from a simple closed curve, how many components does the remainder have? Into how many pieces does n points separate a simple closed curve?

In some respects there are as many, if not more, simple closed curves in the plane of the fourth type shown in Figure 1 as there are triangles, rectangles, and circles. In fact, if there were some way of selecting a simple closed curve at random from the plane, one might consider it odd indeed if the one chosen even contained a straight line segment or the arc of a circle. It does not require much imagination to see how a person who was seeking to learn about all kinds of plane figures would be led to a study of simple closed curves.

Exercises are scattered throughout the article so that the reader who wishes to do more than skim can exercise. Answers are found at the end of the article. Much of this article resembles the chapter entitled *Point Set Topology* from *Insights into Modern Mathematics*, 23rd yearbook of the National Council of Teachers of Mathematics, and is used with permission.

Perhaps you have already felt a need for better explanations for the meaning of such terms as simple closed curve, topologically equivalent, connected, components, separate. It is hoped that the following sections will partially fill this need. The list of terms found at the end of this article may help the reader in locating definitions.

2. TOPOLOGICAL EQUIVALENCE

We mentioned in the first section of this article that a triangle and a "wiggly" simple closed curve are alike in some respects. We say that they are topologically equivalent. A circle, a square, an ellipse, and a triangle are all called simple closed curves.

Topological equivalence. Two sets A, B are topologically equivalent if there is a 1-1 correspondence between them that is continuous both ways.

The curved arc A and the straight line segment B shown in Figure 2 are topologically equivalent. Although there are many 1-1 correspondences that could be chosen, the projection of A onto B provides a convenient one—that is, a point p of A corresponds to the point $f(p)$ of B directly beneath it.

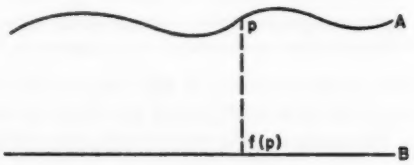


FIG. 2

The 1-1 correspondence from A to B above is continuous because points close together in A go into points that are close together in B. The 1-1 correspondence is continuous the other way because points close together in B correspond to points close together in A. The expression "close together" is not precise and we give a better explanation of continuity in the section *Transformations*.

A 1-1 correspondence of a set A onto a set B that is continuous both ways is called a *homeomorphism* of A onto B. The projection of A onto B is an example of a homeomorphism of A onto B. A homeomorphism of one set onto another set is a special type of transformation and will be discussed further in later sections.

Two sets are *topologically equivalent* if there is a homeomorphism of one onto the other. Accordingly, instead of saying that the two sets are topologically

equivalent, we may say that they are *homeomorphic*. Each set is topologically equivalent or homeomorphic to itself because each point of the set may be made to correspond to itself.

It is possible to get a homeomorphism of a short segment I_1 onto a long segment I_2. (See Fig. 3.) The homeomorphism this time is given by a projection from a point. This homeomorphism shows that, in a certain sense, there are as many points on a short segment as on a long segment.

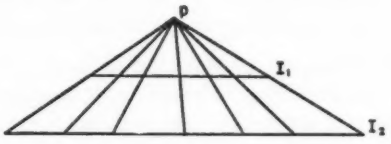


FIG. 3

We mention some other objects that are topologically equivalent or homeomorphic—the surface of a tetrahedron and the surface of a ball; a teacup and a doughnut; the x-axis and the graph of $y=x^2$; the half-open segment* $(0, 1]$ and the graph of $y=1/x$ $(0<x\leqq 1)$. (We suppose that a segment contains its endpoints unless we state otherwise.)

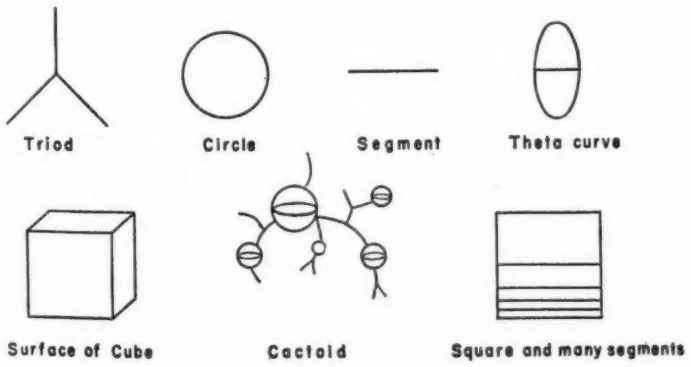


FIG. 4

In Figure 4 we show some objects no two of which are homeomorphic. For example, the circle is not topologically equivalent to the segment because no point of it could be made to correspond (in the proper way) to the end of the segment.

A misconception about topological equivalence. One might suppose that if two sets are topologically equivalent it is possible to deform one onto the other

* By "half-open segment" is meant a segment which contains only one of its endpoints. For example, the half-open segment (0, 1] contains the point 1 but not 0.

by "pulling and stretching but without breaking and tearing." We give some counterexamples to show that this is not always possible.

In E^3 (Euclidean 3-space—the familiar three-dimensional space of everyday experience) consider two sets each consisting of two tangent spheres; in the first set the spheres are tangent externally and in the second set one sphere lies on the interior (except for the point of tangency) of the other. The two sets are topologically equivalent because there is a 1-1 correspondence of the proper sort between the two sets. However, it is not possible to deform one set onto the other in E^3 by "pulling and stretching but without breaking and tearing."

In the next section *Examples of surfaces*, we give other counterexamples to the faulty "pulling and stretching" definition. If one figure can be deformed onto another by pulling and stretching but without breaking and tearing, the figures are topologically equivalent; the converse of this statement is not true. We repeat that two sets are topologically equivalent if there is a 1-1 correspondence between them that is continuous both ways. The definition makes no mention of pulling stretching, and deforming.

Exercise 2. Which pairs of the objects in Figure 5 are topologically equivalent?

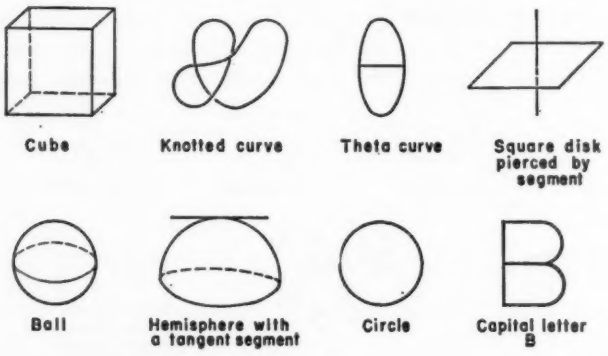


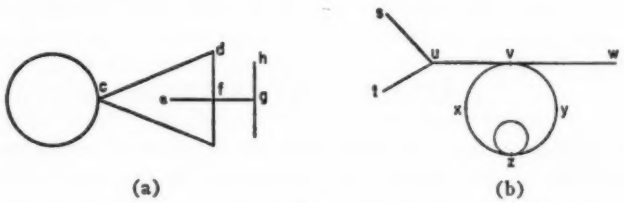
FIG. 5

Exercise 3. Figure 6 shows the capital letters as they may appear without serifs. Group the letters so that letters in the same group are topologically equivalent. Is the grouping different if the letters have width?

A B C D E F G H I
J K L M N O P Q R
S T U V W X Y Z

FIG. 6

Exercise 4. In setting up a 1-1 correspondence between the objects in Figures 7a and 7b to show they are topologically equivalent there are four of the points c, d, e, f, g, h that would necessarily correspond to particular points in Figure 7b, there is one of the six points for which there would be two choices of points in Figure 7b, and there would be another of the six for which there would be many choices. Which points correspond to points c, d, e, f, g, h?



(a) (b)

FIG. 7

3. EXAMPLES OF SURFACES

In this section we give examples of surfaces, some of which are familiar. These examples provide more counterexamples to the faulty "pulling and stretching" definition of topologically equivalent discussed in the preceding section.

Bands. Consider rectangular bands as shown in Figure 8. The arrows in Figure 8a indicate that the side AB of the rectangle is to be "sewed" to CD

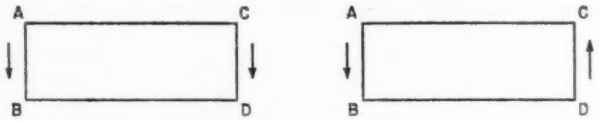


FIG. 8

with A coinciding with C and B with D. Perhaps a cylinder results as shown in Figure 9a but there is no assurance that before the sewing is done, several twists are not made as, say, in Figure 9b, where a full twist is first made. Although the bands in Figure 9a and 9b are topologically equivalent, neither can be obtained from the other by "pulling and stretching without breaking and tearing."

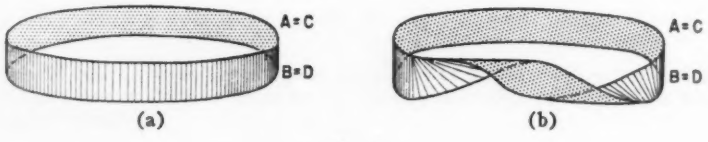


FIG. 9

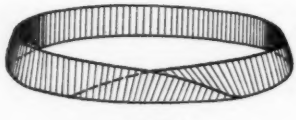


FIG. 10

The arrows in Figure 8b indicate that AB is attached to CD with A coinciding with D and B with C. The resulting band is called a Moebius band. The Moebius band shown in Figure 10 could not be deformed into one with more twists even though the two would be topologically equivalent.

The Moebius band is an example of a one-sided surface. If a fly were to walk around the band an odd number of times, he would be on the "other side" (if there were two sides). Machinery belts are sometimes made into Moebius bands so that they will not wear out on only one side and remain good on the other—they only have one side.

If Moebius bands are split, the results may be unexpected. If it is split down the middle, there results a two-sided band with twists. If this twisted two sided band is split down the middle, there results linking twisted two-sided bands.

The Moebius band has been the butt of jokes and stories. We list several.

1. There are stories of people being given the job of painting one side of a Moebius band and removing all the paint off the other side. See "Alexander Botts and the Moebius Strip," a story by Wm. H. Upson in the December 22, 1945, issue of the *Saturday Evening Post.*

2. One lecturer told the story of how a certain witch doctor kept a supply of bands in the hut in which he gave advice to couples wondering if they were suited to each other. If he wished to prophesy a break-up in their proposed union, he would split an untwisted two-sided band and the two pieces would fall apart. If he wished to prophesy that they would stay together, he would split a two-sided band with a full twist. If he wished to prophesy a really solid marriage, he would split a Moebius band where even after splitting it remained one band. Questioned after his lecture as to the practical value of bands, the lecturer replied, "You might use them if you wanted to be a witch doctor."

3. A three-dimensional poster was used to advertise a certain topology lecture. The poster showed a burlesque dancer whose figure was partially covered by a Moebius band or Moebius strip. The caption was "Would you like to see a Moebius strip? Come to the Auditorium at 8 o'clock tonight."

4. The July, 1949, issue of *Ford Times* carried a story by Wm. H. Upson entitled "Paul Bunyan Versus the Conveyor Belt," telling how Paul Bunyan lengthened a belt in the shape of a Moebius band by splitting it down the middle. Bunyan and his mechanic Fordson won two bets from Loud Mouth Johnson by knowing properties of bands.

5. "No-sided Professor" is the title of an article by Martin Gardner which appeared in the January, 1947 issue of *Esquire.* It tells of the mishaps of a research mathematician who, along with one of his colleagues, disappeared into one of the higher-dimensional spaces while experimenting with odd surfaces.

We mention a more practical use of bands. Suppose a person wishes to study the function $z = y \cos(\pi x/5)$ defined in the strip between the parallel lines $y=1$ and $y=-1$ in the plane. If he divides the strip into rectangles by taking vertical segments on the lines $x=0$, $x=\pm 10$, $x=\pm 20 \cdots$, it may be shown that z takes on the same values on one rectangle as on any other rectangle. Hence the function need be studied on only one rectangle. Also if two points on vertical sides of one of the rectangles have the same y coordinates, z assumes the same values on these two points since

$$y \cos(\pi x/5) = y \cos[\pi(x + 10)/5].$$

Hence we may consider the opposite vertical sides of the rectangle joined to make a cylinder. (See Fig. 11.)

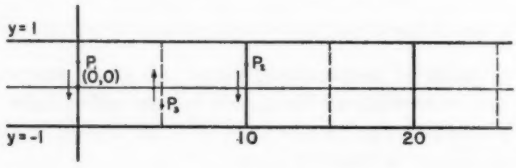


FIG. 11

If one divides the strip into rectangles by taking vertical segments every 5 units rather than every 10 units, two points on opposite vertical sides of one of these smaller rectangles have corresponding z values assigned to them if they are symmetric with respect to the center of the rectangle since

$$y \cos(\pi x/5) = -y \cos[\pi(x + 5)/5].$$

Exercise 5. If the vertical sides of this smaller rectangle are joined so that z takes on the same value on joined points, what figure results?

The large and small rectangles with the two vertical sides joined so that joined points take on the same z values showed why Figure 8a and 8b are useful representations of cylindrical bands and Moebius bands.

Spheres, tori, Klein bottles, and projective planes. If a square as shown in Figure 12a were sewn up as indicated by the arrows, AB would be sewn to AD

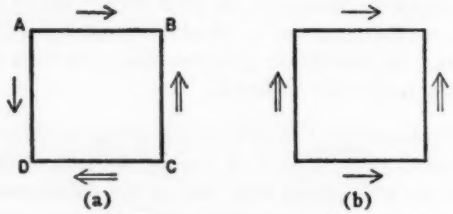


FIG. 12

and CB to CD. There would result a figure that could be distorted into the surface of a ball. A set topologically equivalent to the surface of a ball is called a 2-sphere.

If only the top and bottom of the square shown in Figure 12b were attached, there would result a cylinder. In order to make the rest of the sewing possible in Euclidean 3-space E^3, we suppose that there was no twisting before the first sewing was done. Now suppose the right and left sides are joined as indicated. There are a number of ways that this may be done, three of which are as indicated in Figure 13.

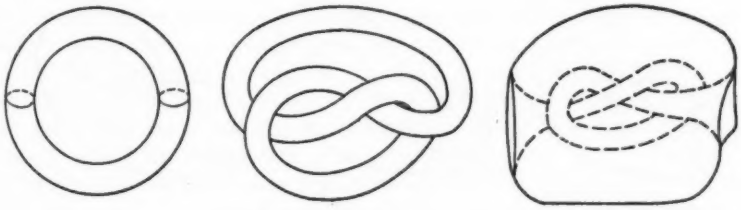


FIG. 13

In Figure 13a, there is no "funny business" and an ordinary *torus* results. In 13b, the cylinder was stretched, a knot was tied in it, and then the final sewing was done. The third is the hardest of the three to visualize. Not only is a knot tied in the cylinder but one end of the cylinder is pulled entirely over the knot before the final sewing is done. If a hole were bored in a cubical block of wood, the surface of the remaining wood would probably be like Figure 13a after certain deformation. If a worm bored a knotted hole in a block of wood, the surface of the remaining wood might resemble Figure 13c. The three surfaces shown in Figure 13 are topologically equivalent to each other but not one of them can be stretched to make it fit on any other one of them. We thus have another counterexample to the faulty pulling and stretching definition.

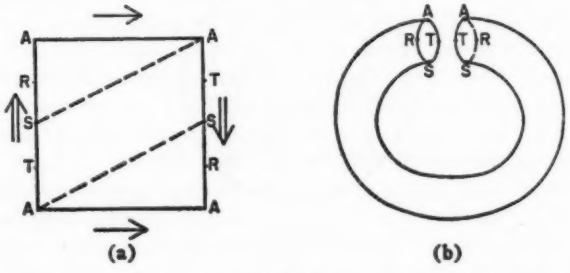


FIG. 14

Not all the objects studied in topology can actually be constructed in E^3. Figure 14 shows such an object. When the first sewing is done a cylinder results as shown in Figure 14b. However, the sewing cannot be completed since the front left arc ARS is to be sewed onto the back right arc ARS and the back left arc ATS is to be sewed onto the right front arc ATS. However we can study objects that we cannot actually construct. If the joining is abstractly made, the resulting surface is called a Klein bottle. If such a surface could be constructed in E^3, it would be a one-sided surface and would have no inside nor outside. However, no such bottle exists in E^3.

FIG. 15

Glass models of Klein bottles appear much as shown in Figure 15a. The left end of a glass tube is run inside the tube, the ends of the tube are joined together on opposite sides of the middle of the tube, and the viewer is to use his imagination and suppose that they miss the middle of the tube where they join. The tube may be split down the middle to make two Moebius bands as shown in Figure 15b. The cut is indicated in Figure 14a by dotted lines.

If a sphere is subdivided into spherical triangles, an order may be chosen on the boundary of each of these triangular disks so that if two of them share an edge in common, the order on this edge is different when viewed from the different disks. For example, we may order the boundaries of the disks clockwise when viewed from the outside of the sphere. In Figure 16, the orientation of

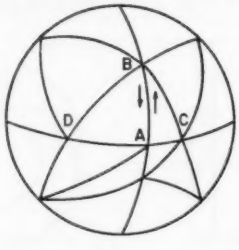


FIG. 16

segment AB is from A to B when considered from spherical triangle ABC but

it is from B to A when considered from disk ABD. Since such an orientation may be chosen, we say that a sphere is *orientable*. Since the rectangle in Figure 14a can be subdivided into small triangles, a Klein bottle can be triangulated into curvilinear disks. However, it is not orientable.

Exercise 6. Suppose the square of Figure 14a is subdivided into 9 squares, each of these squares is divided into two isosceles right triangles, and a clockwise orientation is chosen on the boundary of each triangle. When the sewing is done to get a Klein bottle, where will there be two curvilinear triangles sharing an edge so that the order on this edge is the same from both triangles?

Suppose a top is spinning on the surface of the earth and the top is moved in a path so that it returns to its original position. If the direction of spinning does not change, the top will be spinning in the same direction when it returns. Suppose that instead of having the top reaching out from the surface, we consider a direction of turning (say clockwise) about each point. It is possible to select this direction of turning or orientation so that as a point moves on the surface of a ball the orientation does not change.

The condition is somewhat different on a Klein bottle. Suppose an orientation about a point is chosen and the point is moved the long way around the Klein bottle but so that it returns to its original position. If the orientation is

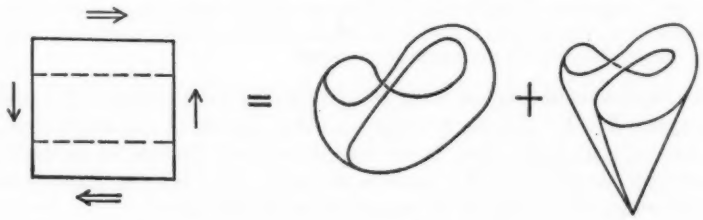


FIG. 17

not changed as the point moves, the orientation will be different at the end from at the beginning. The Klein bottle is not orientable.

The object shown in Figure 17 is a *projective plane*. It is the sum of a Moebius band and a disk. The fitting seems best if the disk is shown as a cone. If it is cut along the dotted lines shown in Figure 17a, the middle portion makes a Moebius band and the outside pieces make a disk. A projective plane is not orientable.

A projective plane is sometimes defined as a circular disk with diametrically opposite points of the boundary connected. The three-dimensional counterpart of a projective plane is projective three space which may be thought of as a solid ball with diametrically opposite points of the boundary joined.

The projective plane is studied in projective geometry. Yet not everyone who studies projective geometry knows of any geometric model for the projective plane.

Exercise 7. Consider a circular disk with a circular hole in the middle as shown in Figure 18. If diametrically opposite points of the outer boundary are joined, what figure results?

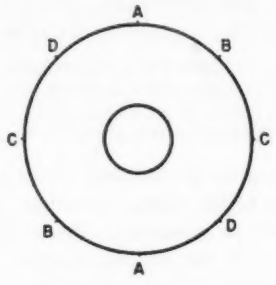


FIG. 18

Exercise 8. Suppose the function $z=\cos x \cos y$ is defined on the horizontally based square in the plane with opposite vertices (0, 0) and (π, π). What figure is formed if opposite vertical sides are sewed together so that z takes on the same value on joined points? If the top and bottom are also joined in this fashion, what figure is formed?

Exercise 9. Describe a function $z=f(x, y)$ that can be defined on a horizontally based square with opposite vertices (0, 0) and (1, 1) so that studying z on the square is equivalent to studying it on a Klein bottle after points of the boundary are joined.

The *complement* of a figure is the set of points not in the figure. (Complement of A in space S is $S-A$.) For example, on the surface of the earth the surface of the oceans, seas, lakes, etc., is the complement of the surface of the land. In the plane the complement of a circle is the sum of its interior and its exterior. The complement of a circle in E^3 would be all in one piece.

We now show that even though two figures in the same space are topologically equivalent, their complements may be topologically different.

In Figure 13, each of the surfaces has an interior and an exterior. It could be shown that the interior of the first is topologically equivalent to the interior of the second but not to the interior of the third. A small creature who knew about topological equivalence but not about straightness, length, etc., could not tell the difference between the inside of the first tube and the interior of the second. However, the exterior of the first tube in Figure 13 is like the exterior of the third but not like the exterior of the second. No two of the three figures have complements that are topologically equivalent.

The knotted curve shown in Figure 5 is topologically equivalent to a circle but their complements are topologically different.

Examples and counterexamples. Although examples are not the main forte of topology, they do serve a useful purpose. A wealth of examples helps our intuition in picking out certain theorems that are likely to be true and labeling others as false.

Examples may be used as counterexamples to certain conjectures and show

that it is useless to try to prove them. A counterexample is a convincing way of showing that a "theorem" is, in fact, false. We know in this way that each of the following conjectures is a false statement.

FALSE STATEMENT A. *If two objects in E^3 are topologically equivalent, it is possible to deform one onto the other with pulling and stretching without breaking and tearing.*

Exercise 10. Name 3 pairs of topologically equivalent objects that serve as counterexamples to False Statement A.

FALSE STATEMENT B. *Each surface is orientable.*

Exercise 11. Name three counterexamples to False Statement B.

FALSE STATEMENT C. *Each surface is two-sided.*

Exercise 12. Name a counterexample to False Statement C.

FALSE STATEMENT D. *If two surfaces in E^3 are topologically equivalent, their complements are topologically equivalent also.*

Exercise 13. Give a counterexample to False Statement D.

4. LIMIT POINTS

The notion of a limit is a very important one in mathematics. A student may do well in arithmetic, algebra, and even geometry without understanding limits, but he must learn this concept in order to go far in mathematics. Although limits are used extensively in calculus, some students do the mechanical parts of this subject without gaining an insight into the concept of limits. Limits are used so extensively in point set topology that it is inconceivable that a person could make much headway here without learning a considerable amount about limits. Learning about limit points is a good starting point.

We use the notion of a limit to tell us whether a point is "infinitely close" to a set—whether it "sticks onto" a set. In such a nice space as the Euclidean plane we use the idea of distance to define limit points. In the more general spaces such as those discussed in the section *Some examples of abstract spaces* we use neighborhoods to define limit points.

NEIGHBORHOOD. *A neighborhood in the Euclidean plane E^2 is the interior of a circle. If a point lies on the interior of a circle, this interior is called a neighborhood of the point.*

Each point has many neighborhoods,—some small and some large, some with the point in the middle and some asymmetric with respect to the point. Each neighborhood is a neighborhood of many points—in fact, of each point in it.

A neighborhood in Euclidean 3-space E^3 is the interior of a sphere; in Euclidean 1-space E^1 (or a line) it is an open segment (segment without its end points)—in fact, in any space with a distance, it is the interior of a generalized sphere. In more general spaces such as those discussed in the section *Some examples of abstract spaces*, there may be no distance function and the neighborhoods are certain designated point sets that satisfy certain conditions or axioms for these abstract spaces. A distance function is defined in the section *Some examples of topological spaces* but our intuitive notion of how far things are apart will suffice here.

NEIGHBORHOOD DEFINITION OF LIMIT POINT. *The point p is a limit point of the set X if each neighborhood of p contains a point of X other than p.*

(Note that a limit point of X is not necessarily a point of X.)

DISTANCE DEFINITION OF LIMIT POINT. *The point p is a limit point of the set X if for each positive number ϵ there is a point x of X such that $0<\rho(p, x)<\epsilon$.*

(We use $\rho(p, x)$ to denote the distance from p to x.) This definition is not applicable in abstract spaces without distance functions.

Exercise 14. Suppose that, in the plane, X is the set of all points satisfying the condition $0<x^2+y^2<1$ or $x=y=1$. Which of the following points are points or limit points of X? $p=(0, 0)$, $q=(0, 1)$, $r=(1, 1)$, $s=(2, 0)$, $t=(0, \frac{1}{2})$.

Exercise 15. In an abstract space whose points are the points of the plane and whose neighborhoods are horizontal lines, which of the points p, q, r, s, t are limit points of the set X of Exercise 14?

Sets. A *point set* is a collection, each of whose members (or elements) is a point. The collection of all points with a particular property R is designated by $\{p \mid p \text{ has property } R\}$. A point set with only one point p is designated by $\{p\}$. We use the following terminology in discussing sets.

SUBSET. *We say that A is a subset of B if each element of A is an element of B.*

We denote this by $A \subset B$. If p is an element of A, we write $p \in A$. We may say that p belongs to A.

SUM OR UNION. *The sum of A and B (denoted by $A+B$ or $A \cup B$) is the set of all elements in either A or B—that is $A+B=\{p \mid p \in A \text{ or } p \in B\}$.*

INTERSECTION, PRODUCT, OR COMMON PART. *The intersection of A and B* (denoted by $A \cdot B$, $A \cap B$, *or* AB) *is the set of all elements in both A and B—that is, $A \cdot B=\{p \mid p \in A \text{ and } p \in B\}$.*

If there is no element which is in both A and B, we say that A does not intersect B and may write $A \cdot B=0$.

Some people use the artificial convention that a collection need have no elements. This may lead to complications in having to make exceptions for the null set, as this artificial thing is called. For example, instead of saying that two parallel lines do not intersect, some of those who use this "null set" convention say that two parallel lines meet in the "empty set." Instead of saying that each bounded collection of numbers has a least upper bound, they say that each nonnull bounded collection of numbers has a least upper bound. The definition of "connected" as given in the section *Properties preserved by mappings* would contain an extra condition if the "null set" convention were used. On the other hand, the convention may have some advantage in the algebra of sets (Boolean algebra) for then two sets always have a product and a difference. We believe the disadvantages of introducing the null set in our treatment of point set topology outweighs the advantages so we do not use it here. Anytime we say that X is a set, we imply that there is at least one element in X.

Not only do we add and multiply two sets but we do the same for three sets, four sets ... or even an infinite collection of sets. Suppose $\{A_\alpha\}$ is a collection (perhaps infinite) of point sets and A_α is a typical set in $\{A_\alpha\}$. The sum (or union) of the elements of $\{A_\alpha\}$ is the set of all points p such that each belongs to some A_α. We use $\sum A_\alpha$ or $\cup A_\alpha$ to denote this sum. Then $\sum A_\alpha = \{p \mid p \in \text{some } A_\alpha\}$. The product, intersection, or common part of the elements of $\{A_\alpha\}$ may be written $\prod A_\alpha$ or $\cap A_\alpha$. Then $\prod A_\alpha = \{p \mid p \in \text{each } A_\alpha\}$. If there is no point common to all elements of $\{A_\alpha\}$ we say that this intersection does exist and write $\prod A_\alpha = 0$.

DIFFERENCE. *We use $A-B$ to denote the collection of elements of A that do not belong to B—that is, $A-B=\{p \mid p \in A \text{ and } p \notin B\}$. If $A \subset B$, we write $A-B=0$.*

An example. If a person were to squirt some black ink on the plane, the set A of points in the dark spot is an example of a point set. Suppose a set B is determined by squirting some red ink on the plane. Then $A+B$ designates the set of points covered by ink, $A \cdot B$ designates the set covered by both kinds of ink, and $A-B$ designates those covered by black but not red ink. (See Fig. 19.)

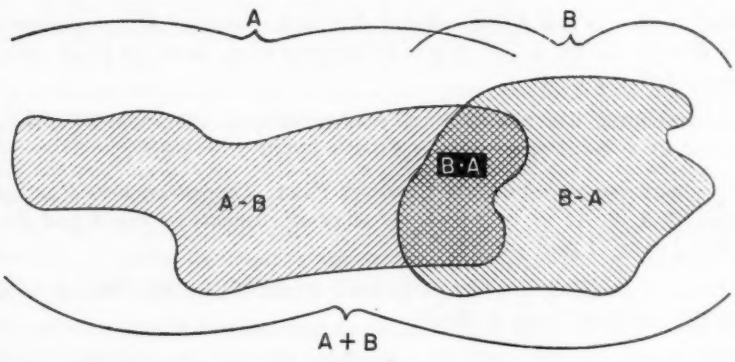


FIG. 19

OPEN SET OR DOMAIN. *A point set D is an open set if for each point p of D there is a neighborhood of p that lies in D. An open set is called a domain. The intersection of a set X and an open set is called an open subset of X.*

(Note that an open subset of X may not be an open set.)

An open set is the sum of neighborhoods, although they may be infinite in number. For the Euclidean plane, the interior of a square is a domain but a disk (circle plus its interior) is not.

Exercise 16. In showing that the sum of two open sets A, B is open, one shows that if p is a point of $A+B$, p belongs in a neighborhood which lies in $A+B$. There are two cases in the proof—either p belongs to A or it does not. If it does not belong to A, show that there is a neighborhood N of p in $A+B$ containing p.

Exercise 17. Suppose S is the interior of a square in the plane. For each point p of S describe a radius r_p such that if $V(p, r_p)$ is the interior of a circle with center at p and radius r_p, then $\sum V(p, r_p) = S$.

CLOSED SET. *A closed set is one which contains all its limit points. Each of its limit points belongs to it. A line is closed in the plane.*

The *closure* of a set X is the sum of X and all its limit points. This closure is designated by $\overline{X}$. The closure of the set X of Exercise 14 is the sum of a disk and $\{r\}$. The closure of the set X of Exercise 15 is the set of all points above the line $y=-1$ and on or below the line $y=1$.

Exercise 18. For the Euclidean plane give an example of (a) a closed set, (b) an open set, (c) a set neither open nor closed, and (d) a set that is both open and closed.

Exercise 19. What would need to be proved to show that the closure of each set is closed?

Exercise 20. If, in the plane, G is the graph of $y=\sin 1/x$ $(0<x<1/\pi)$, what is $\overline{G}$?

It is possible to prove each of the following theorems from the definitions alone without even knowing what a neighborhood looks like:

1. *The sum of two open sets is open.*
2. *The product of two closed sets is closed.*
3. *The complement of a closed set is open.*
4. *The complement of an open set is closed.*
5. *The closure of each set is closed.*

The reader may gain experience by proving these from the definitions. Exercises 16, 21, 22, 23, 24 point in the direction of these proofs. The first two theorems may be generalized to say that the sum of any collection (perhaps infinite) of open sets is open and the product of any collection of closed sets is closed.

Exercise 21. In showing that the product of two closed sets A, B is closed, one considers a limit point p of $A \cdot B$ and shows that p is a point of $A \cdot B$. Show that if p is a limit point of $A \cdot B$, it is a point of A.

Exercise 22. If A is a closed set and p is a point of its complement, show that the assumption that there is no neighborhood of p that misses A leads to the contradiction that $p \in A$.

Exercise 23. If A is an open set and A' is its complement, show that if $p \notin A'$, then p lies in a neighborhood N that misses A'.

Exercise 24. Show that if A is a set and $p \notin \bar{A}$, then p is not a limit point of $\bar{A}$.

5. TRANSFORMATIONS

Ordinarily, a person regards a transformation as a change. A seed becomes a tree, baby becomes an adult, wood turns to smoke and ashes, a bomb changes to fragments, a wad of gum is squeezed, one object is transformed into another.

In mathematics a person may come to regard a transformation as a rule or law associating points of one set X with points of another set Y. If f denotes the transformation and $x \in X$, we use $f(x)$ to denote the point of Y associated with x. (See Fig. 20.)

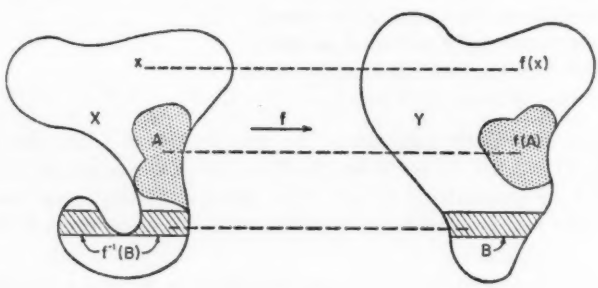


FIG. 20

We call $f(x)$ the *image* of x under f. If A is a subset of X, $f(A)$ denotes the collection of images of points of A. If B is a subset of Y, $f^{-1}(B)$ is the set of all points of X that go under f into a point of B; $f^{-1}(B)$ is called the *inverse image* of B.

A useful and more sophisticated approach is to define a transformation of X into Y to be a collection of ordered pairs that satisfy certain conditions. For example, if one considers the equation $y = x^2$, the corresponding transformation might be regarded as the set of all ordered pairs of the form (a, a^2). The pairs would be the coordinates of points on the parabola. A definition of a transformation based on ordered pairs may seem somewhat artificial and perhaps not exactly describing the intuitive notion that we have in mind, but it does have the advantage of being a precise definition given in terms of more elementary concepts.

ORDERED PAIRS DEFINITION OF TRANSFORMATION. *A transformation of X into Y is a collection of ordered pairs (x, y) such that $x \in X$, $y \in Y$, and each element of X is the first element of one and only one of the ordered pairs.*

FUNCTION DEFINITION OF TRANSFORMATION. *We may regard a transformation of X into Y as a single-valued function that assigns one and only one element of Y to each element of X.*

This definition leaves it to those who define function to decide if it is a change, a rule, a collection of ordered pairs, or something else.

If each element of Y is the image of an element of X under f, we say that f takes X *onto* Y instead of merely into Y. Hence, of the two transformations of the reals into the reals given by $y=x^2$ and $y=x^3$, the second is an onto transformation but the other one is not.

If no element of Y is the image of two elements of X, we say that f is *one-to-one* or 1-1. Hence $y=\sin x$ is 1-1 for $0<x<\frac{1}{2}\pi$ but not for $0<x<\pi$.

Intuitively, we say that a transformation f is continuous if points "close together" in X go into points "close together" in Y. The notion "close together" is not precise so we give the following definition:

NEIGHBORHOOD DEFINITION OF CONTINUOUS. *A transformation f of X into Y is continuous at the point x of X if for each neighborhood U of $f(x)$ there is a neighborhood V of x such that $f(V \cdot X) \subset U$.* (See Fig. 21.)

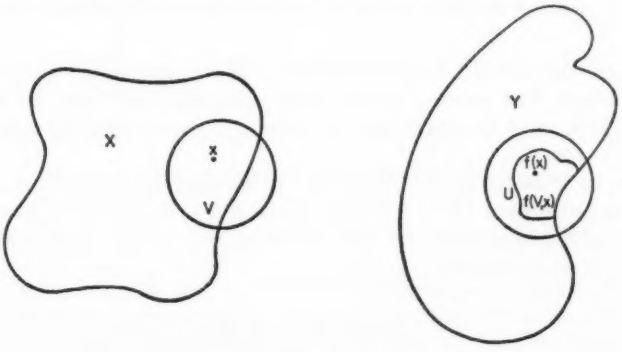


FIG. 21

The size of V in the above definition is a function of x and U—if U is taken to be small, V may need to be chosen small. However, Example 8 below shows that V may not need to be small even if U is.

If the set X on which f is defined is all of space, the definition may be simplified by replacing "$f(V \cdot X) \subset U$" by "$f(V) \subset U$." We could make this change anyway if we called V an open subset of X containing x rather than a neighborhood of x.

The set on which a function is defined is called the *domain of definition* of

the function. Note that a domain of definition may not be an open set or a domain as defined earlier.

A *function is continuous* if it is continuous at each point of its domain of definition X. A continuous function is called a *map*.

For metric spaces (Euclidean spaces and other spaces with distance functions), we may use the following definition:

$\epsilon-\delta$ DEFINITION OF CONTINUITY. *The transformation f of X into Y is continuous at the point x of X if for each positive number ϵ there is a positive number δ such that for each point x' of X within a distance δ of x, $f(x')$ is within ϵ of $f(x)$.*

Let us consider some examples of transformations of the reals into the reals described by equations.

1. $f(x)=x^3-1$.
2. $f(x)=x^3-x$.
3. $f(x)=x^2$.
4. $f(x)=\sin x$.
5. $f(x)=\text{Arc tan } x$.
6. $f(x)=0$ if x is irrational or 0, $f(p/q)=1/q$ if $p/q\neq 0$ is rational and in lowest terms.
7. $f(x)=x$ or x^3 according as x is algebraic or nonalgebraic.
8. $f(x)=1-x^2$, 0, or x^2-1 according as $x<-1$, $-1\leqq x\leqq 1$, $x>1$.

Exercise 25. Which of the above examples is (a) onto, (b) 1-1, (c) continuous, (d) continuous at $x=0$?

Here is an example of a homeomorphism f of the interior of a circle of radius 1 onto the plane. The center c of the circle goes into itself and if p is another point of the interior of the circle, $f(p)$ is on the ray from c through p and $\rho(c, f(p))$ is $\rho(c, p)/(1-\rho(c, p))$.

Now let us consider a transformation of one segment AC onto a segment $C'B'$ as shown in Figure 22. The points between A and B go into points of $A'B'$ through parallel projections and the points of BC go into points of $B'C'$ by projections through a point.

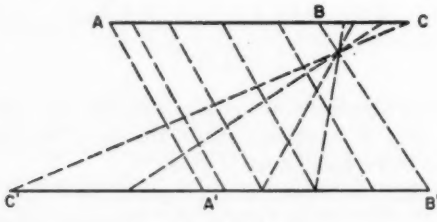


FIG. 22

If P is a point of the segment AB, the ratio in which P divides AB is the

quotient obtained by dividing the distance from A to P by the distance from P to B. The map from AB to $A'B'$ is called *linear* if for each point P of AB, P' divides $A'B'$ in the same ratio that P divides AB. We note that while the map shown in Figure 22 is linear on AB and is linear on BC, it is not linear on AC. We make use of linear maps in the section *The Jordan curve theorem.*

This transformation indicated by Figure 22 may be represented by the graph shown in Figure 23. The interval AC is placed on the x-axis and the interval $C'B'$ on the y-axis. To find the image of a point of AC, project the point vertically onto the graph and then project this point horizontally onto the y-axis. The equation in this case is $f(x)=x+2$ or $17-4x$ according as $1\leqq x\leqq 3$ or $3\leqq x\leqq 4$.

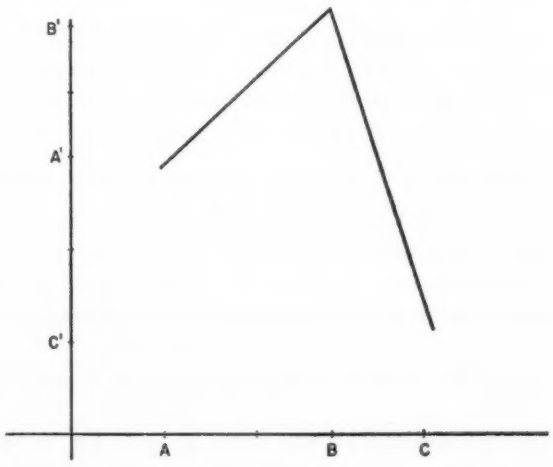


FIG. 23

If X is a square plus its interior and Y is a segment, a transformation of X into Y may be represented by a graph in 3-space which may be a surface. However, if Y were also a square plus its interior, it would require a space of four dimensions to draw the graph—two dimensions for X and two others for Y.

The $\epsilon-\delta$ definitions of continuity of functions given in analytics and calculus texts are puzzling to most students when they are encountered for the first time. In some cases, even an expert could not tell from the definitions given whether or not a function such as $f(x)=\sqrt{(1-x^2)}$, $-1\leqq x\leqq 1$, is continuous at $x=1$ since mention of the domain of definition is sometimes omitted—in other cases the function would be called discontinuous because it is not defined for all values of x. From the definitions we have given, the corresponding transformation would be continuous.

Here is a geometric definition of the continuity of a graph which may be

more meaningful to many students than the corresponding $\epsilon-\delta$ definition. It is understood that the graph is in the plane.

DEFINITION OF CONTINUITY OF A GRAPH. *The graph G is continuous at the point p of G if for each pair of horizontal lines, one above and one below p, there is a pair of vertical lines, one to the right and one to the left of p, such that each point of G between the vertical lines is between the horizontal lines.* (See Fig. 24.)

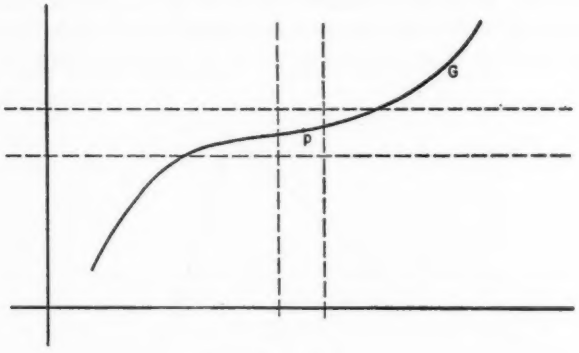


FIG. 24

Exercise 26. What in the above definition corresponds in the $\epsilon-\delta$ definition to (a) x, (b) $f(x)$, (c) ϵ, (d) δ?

Exercise 27. Under what conditions would a graph in 3-space be continuous at one of its points p?

The definition of topologically equivalent given in the section *Topological equivalence* may have been unsatisfactory since no precise definition of continuous had been given. This flaw can now be removed.

A 1-1 transformation f of A onto B is said to be ***reversibly continuous*** if f^{-1} is continuous at each point of B.

A 1-1 transformation f of A onto B is called a ***homeomorphism*** of A onto B if it is continuous and reversibly continuous.

Two sets are said to be *topologically equivalent* (or *homeomorphic*) if there is a homeomorphism of one onto the other.

6. PROPERTIES PRESERVED BY MAPPINGS

A continuous transformation is called a *mapping*. In this section we point out that certain properties are preserved by mappings. The following exericse shows that in a certain sense, limit points are preserved.

Exercise 28. Show that if p is a limit point of X and if f is a map of $\overline{X}$ into Y, $f(p)$ is either a point or a limit point of $f(X)$.

Suppose $x_1, x_2, \cdots$, is an infinite sequence of points—we do not suppose that different elements of the sequences are different points. The sequence is said to *converge* to a point x_0 if for each neighborhood U of x_0 there is an integer n such that each point of $x_n, x_{n+1}, \cdots$ lies in U. A proof similar to that used to answer Exercise 28 shows that if f is a mapping of X, X contains $x_0, x_1, x_2, \cdots$, and $x_1, x_2, \cdots$ converges to x_0, then $f(x_1), f(x_2), \cdots$ converges to $f(x_0)$.

The preceding result may remind us of the definition of continuity of a function given in some calculus books. The function $f(x)$ is sometimes said to be continuous at $x=a$ provided the limit, as x approaches a, of $f(x)$ is $f(a)$.

Connectedness is another property that is preserved by mappings. Intuitively, we think that a thing is connected if it is "all in one piece." For example, the disk D shown in Figure 25 is connected but the Cantor set C is not. However, D is the sum of its interior and its boundary, so in a certain sense it is in

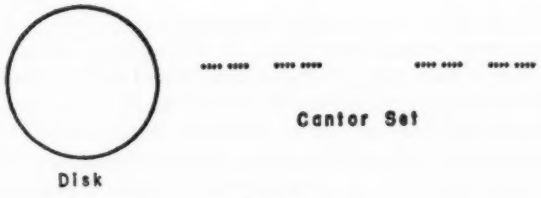


FIG. 25

two pieces. (The *boundary* of M is the intersection of $\overline{M}$ and the closure of the complement of M.) We shall give a more precise definition of connected. The

Cantor set shown in Figure 25 is obtained by starting with a segment, removing the open middle third of it, removing each of the open middle thirds of the remaining pieces, etc.

Two sets are *mutually separated* if neither contains either a point or a limit point of the other. (A, B mutually separated means $\overline{A}\cdot B = A\cdot\overline{B} = 0$.) The interior of a disk and its boundary are not mutually separated since a point of the boundary is a limit point of the interior.

A set is *not connected* if it can be expressed as the sum of two mutually separated sets. The Cantor set is not connected because its right-hand half and left-hand half are mutually separated.

DEFINITION OF CONNECTED. *A set is connected if it is not the sum of any two mutually separated sets.*

Exercise 29. Show that if f is a mapping of a connected set X, then $f(X)$ is connected also.

A closed and connected point set is called a continuum. Being a continuum is not preserved under a mapping—we showed in the last section a homeomorphism of the interior of a circle (not a continuum) onto the plane (a continuum). The inverse of this homeomorphism is a mapping of a continuum onto something that is not closed.

A *component* of a set X is a maximal connected subset of X. Hence, the graph of a hyperbola has two components but the graph of a parabola has only one. The complement of a hyperbola in the plane has three components while the complement of a parabola has only two. A continuum has only one component but a Cantor set has infinitely many.

Being open and closed is preserved in a certain sense by the inverses of mappings. In fact, rather than say that a transformation f of X into Y is continuous if it is continuous at each point of X, it is sometimes defined to be continuous if the inverse of each open subset in Y is an open subset of X.

Exercise 30. Show that if f is a continuous transformation of X onto Y, then the inverse of each open subset of Y is an open subset of X.

Exercise 31. Show that the transformation f of X into Y is continuous if the inverse of each open subset of Y is an open subset of X.

Exercises 30, 31 show that a transformation f of X into Y is continuous if and only if the inverse of each open subset of Y is an open subset of X. Similarly it might have been shown that f is continuous if and only if the inverse of each closed subset of Y is a closed subset of X. (A closed subset of X may not be closed but is merely the intersection of X with a closed set.)

As the population of the earth increases, it is inevitable that there be centers of dense population. In contrast, it is felt that if space travel becomes feasible and space is habitable, the population of the universe can increase without us having crowding. Even in the plane, it is possible to have infinitely many points such that no point is within 100 units of any two of them. (Put the points at (0, 0), (300, 0), (600, 0), · · · .) The limited property of the earth we have men-

tioned reminds us of a topological property called compactness. Compactness is preserved under a mapping if we use the following definition.

DEFINITION OF COMPACTNESS. *A set M is compact if each infinite subset of M has a limit point in M.*

The plane is not compact but, as shown in the next section, a closed segment is. (Some definitions do not impose the condition that the limit point of M need belong to M. If this condition were omitted, the interior of a circle would be compact and Exercise 32 could not be done. Other definitions impose even more stringent conditions.)

Exercise 32. Show that if f is a mapping of X and X is compact, then $f(X)$ is compact also.

A set of tarpaulins is said to cover a football field if each spot of the field is under one of them. Similarly, a collection G of point sets is said to *cover* a point set X if each point of X belongs to an element of G. If each element of G is a domain, G is called an *open covering*.

The collection of all neighborhoods covers the plane but no finite subcollection of them does. However, if a collection of neighborhoods covers the surface of a sphere or a segment, a finite number of them does also. This reminds us of the following topological property that is preserved under a mapping.

DEFINITION OF BICOMPACTNESS. *A set X is bicompact if each open covering G of X contains a finite subcovering—that is, a finite subcollection of G which covers X.*

Exercise 33. Show that bicompactness is preserved by a mapping.

The two properties, compactness and bicompactness, are not equivalent because there are topological spaces that are compact but not bicompact. However, any bicompact space is compact and any compact space with a distance function is bicompact. Therefore, bicompactness is a stronger condition and a person who uses it in the hypothesis of a theorem is not proving as strong a result as one who uses mere compactness instead. Because of lack of agreement on terminology, some people use the term compact to mean bicompact as we have defined it here. Not everyone that mentions the word compact is using the weaker condition.

Exercise 34. Show that bicompactness implies compactness.

A set X is *homogeneous* if for each pair of points p, q of X there is a homeomorphism of X onto itself that takes p onto q. A circle is homogeneous because a rotation will take any point into any other point. A closed segment is not homogeneous because there is no homeomorphism of the segment onto itself that takes an end point onto a non-end point.

A definition of topology. Topology may be defined as a study of properties that remain invariant under homeomorphisms. Such a definition might be meaningless to persons unacquainted with such properties as connectedness,

compactness, bicompactness, homogeneity, etc. If one were to give such a definition to an unenlightened gathering, the question might be asked as to what is a homeomorphism. If a definition of "homeomorphism" were given such as appears in this article, the next question might be, what is the meaning of the word "continuous"? Neither the neighborhood nor the $\epsilon-\delta$ definition of continuity is likely to be of much help. Perhaps it is best not to define topology to people who do not already know what it is.

Unsolved problems. A difficulty in discussing unsolved problems in topology is that most of these problems involve complicated concepts meaningful only to the advanced worker. Most people would have neither the technical vocabulary nor the experience to appreciate the discussion. However, here is a problem which we have at least developed the vocabulary to present.

A *simple closed curve* is a set topologically equivalent to a circle. Is a simple closed curve the only homogeneous compact plane continuum other than a point? For many years it was conjectured that the answer was in the affirmative—in fact some papers appeared in scientific journals giving faulty proofs to this effect. However, in 1948, an example was given of a bounded plane continuum which was neither a simple closed curve nor a point but which was homogeneous nevertheless. In 1954 another example was announced. Are there other homogeneous bounded plane continua?

7. THE JORDAN CURVE THEOREM

The interior of circles and triangles appear to be somewhat alike. However, if one has a complicated simple closed curve such as shown in Figure 26, one might have difficulty in deciding if its interior was like the interior of a circle.

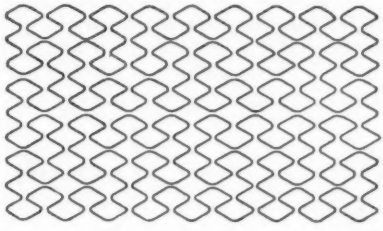


FIG. 26

If the curve had been even more complicated, it might have been impossible to tell by merely looking whether or not it even had an interior.

A subset X of a set Y is said to *separate* Y if Y is connected but $Y-X$ is not connected. It may seem intuitively obvious that a simple closed curve in the plane separates the plane but it is not easy to prove this.

The *Jordan curve theorem* states that the complement of each simple closed curve J in the plane has exactly two components and is the boundary of each of them. One extension of the Jordan curve theorem due to Schoenflies may be stated as follows:

AN EXTENSION OF THE JORDAN CURVE THEOREM. *If J_1 and J_2 are simple closed curves in the plane, there is a homeomorphism of the plane onto itself that takes J_1 onto J_2.*

In plane analytical geometry one studies translations and rotations of the plane onto itself—and perhaps even reflections through a line. These are examples of homeomorphisms of the plane onto itself. Under each of these particular homeomorphisms, a figure would go into a figure congruent to itself. However, the types of homeomorphisms used to show the truth of the above extension of the Jordan curve theorem would frequently change the shapes of objects.

To those who think the theorem needs no proof since it is intuitively obvious, it may come as a surprise that the analogous theorem is not true in 3-space. Although the horned sphere described at the end of this section is topologically like the surface of a ball, there is no homeomorphism of 3-space onto itself that will take one onto the other.

The Jordan curve theorem is one of the more frequently used theorems of plane topology; it is used in such subjects as analysis or complex variables. Although several proofs of this theorem have been given—and some only recently—these proofs are still complicated.

In this section, we only treat the Jordan curve theorem where the simple closed curves involved are uncomplicated. However, if the reader works through even the first of the exercises, he will get some understanding of the meaning of the theorem.

Exercise 35. Describe a homeomorphism of the plane onto itself that takes a given triangle onto a given circle. (See Fig. 27 for a hint as to how to proceed.)

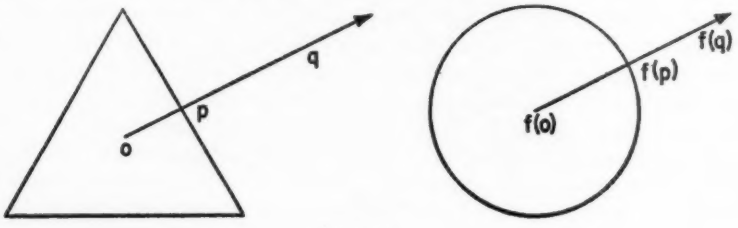


FIG. 27

Exercise 36. Describe a homeomorphism of the plane onto itself that takes a given triangle onto the graph of $\{y-\sqrt{(1-x^2)}\}\{2y-\sqrt{(1-x^2)}\}=0, (-1\leqq x\leqq 1)$.

By using mathematical induction on the number of sides, it can be shown that for any polygon there is a homeomorphism of the plane onto itself that takes the polygon onto a triangle. One without training in topology who obtained an airtight proof of such a theorem could be proud of his accomplishment. Indeed, it is no easy task for the beginner to get even Exercise 37. Some of the methods used in working the following exercise can be used to get the more general result.

Exercise 37. Describe a homeomorphism of the plane onto itself that takes polygon *ABCDEF* of Figure 28 onto a triangle. (See Fig. 29 for a hint as how to proceed.)

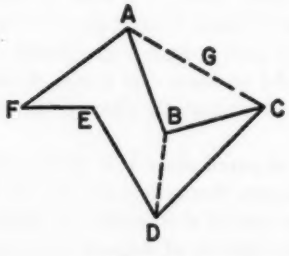


FIG. 28

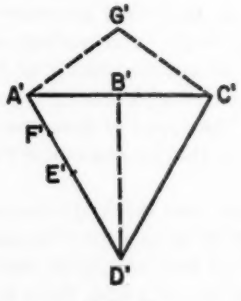


FIG. 29

A set in the plane is *bounded* if it lies on the interior of some large circle. The interior of a triangle in the plane is bounded but its exterior is not.

It is to be noted that we regard a triangle as the sum of its three sides. The complement of a triangle in the plane has two components. If ABC is a plane triangle we use Int ABC to denote the bounded component of the complement of ABC. Either the Jordan curve theorem or the Schoenflies theorem shows that each simple closed curve in the plane has a complement with a bounded component.

The reader who is more interested in getting an overall view of this chapter than in going into details may skip the rest of this section and proceed with the next section *Properties of the real numbers.*

We now give three theorems relating to the Schoenflies theorem. Each is of interest in its own right.

RETRACTION THEOREM. *Suppose P is a polygon and ABC is a triangle such that $P\cdot(ABC+\text{Int } ABC)$ is the sum of two sides AB, BC of ABC. If AC is the third side of ABC, there is a homeomorphism h of the plane onto itself that takes P onto $P-(AB+BC)+AC$.*

Let M be the midpoint of AC and R, S be the points on the line MB as shown in Figure 30 so that P intersects the disk bounded by the polygon $ARCS$

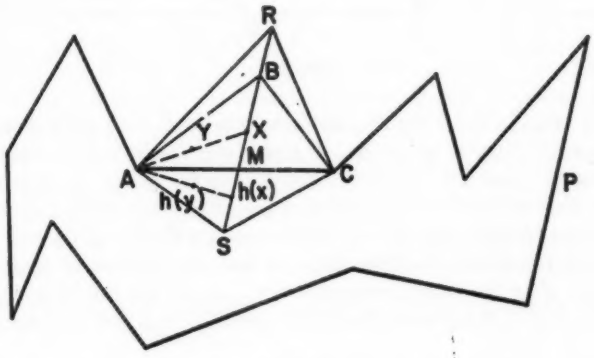


FIG. 30

only in $AB+BC$. The homeomorphism h is the identity outside the disk, takes RB linearly onto RM and BS linearly onto MS.

Exercise 38. If X is a point of RS and Y is a point of Int ARS between A and X, how might h be defined on Y?

SCHOENFLIES THEOREM FOR POLYGONS. *For each polygon P in the plane there is a homeomorphism h of the plane onto itself that takes P onto a triangle.*

SHELLING THEOREM. *If P is a polygon with more than three sides and UV is a side of P, there is a triangle ABC such that AB, BC are sides of P different from UV, and $(ABC+\text{Int } ABC)$ lies, except for $AB+BC$, on* Int P.

These two theorems may be proved by induction on the number $n(P)$ of sides of P by making use of the retraction theorem. Each holds for $n(P)=3$ or 4 so we assume $n(P)=k$ is the first integer for which there is a polygon P for which one of the two results does not hold.

Let L be a line of support for this P (P intersects L but not both components of the complement of L) such that L contains two points of P. Since the theorems would be easier to prove if P were assumed to be a convex polygon, we suppose that L contains a segment AB that intersects P exactly in its end points. (See Fig. 31.)

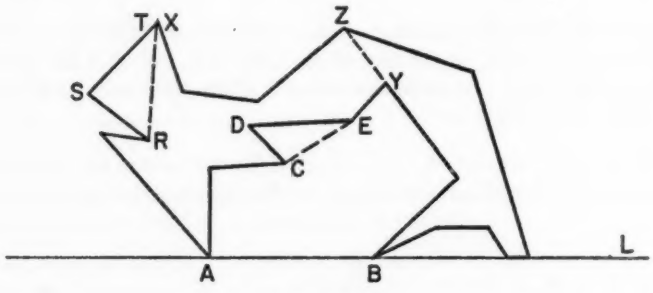


FIG. 31

Let AXB and AYB be the two broken lines in P with ends A and B. With no loss of generality we suppose that X and Y are vertices of P chosen to be at a maximum distance from L and such that X is as far if not farther from L than Y. If X is farther than Y from L, we find that there is a vertex Z of AXB such that Z is farther from L than Y and YZ intersects P only at its end points. If it should happen (which it does not, but we are not yet ready to prove that it does not) that X is the same distance from L as Y, we would select Y so that there is a vertex Z of AXB such that YZ is parallel to L and intersects P only at Y and Z.

We now show that the Schoenflies theorem is satisfied on P. Consider the polygon $AB+AYB$. If it is a triangle, it follows from the retraction theorem that the Schoenflies theorem is satisfied on P. If $AB+AYB$ has more than three sides, the following five exercises indicate why the Schoenflies theorem holds on P.

Exercise 39. Why is the shelling theorem satisfied on $AB+AYB$?

Exercise 40. In applying the shelling theorem to $AB+AYB$ to show that there is a triangle CDE such that CD and DE are sides of P other than UV, and CDE lies, except for $CD+DE$, on Int $(AB+AYB)$, what should be used for UV to insure that CD, DE are sides of P?

Exercise 41. Why will not the segment CE of Exercise 40 intersect P at a point other than C and E?

Exercise 42. Name a polygon P' with one fewer sides than P such that the retraction theorem says that there is a homeomorphism of the plane onto itself taking P onto P'.

Exercise 43. If h is a homeomorphism of the plane onto itself taking P onto P' and g is a homeomorphism of the plane onto itself taking P' onto a triangle, name a homeomorphism of the plane onto itself that takes P onto a triangle.

Finally we consider why the shelling theorem is satisfied on P. Let $ZA\,Y$ and $ZB\,Y$ be the two arcs whose sum is P. If UV is a given arc of P, we choose the one of $ZA\,Y$, $ZB\,Y$ that does not contain it. With no loss of generality we suppose it is $ZA\,Y$. The shelling theorem is satisfied on polygon $ZA\,Y+YZ$ since this polygon has fewer sides than P. It follows from this shelling theorem that there is a triangle RST such that RS, ST are sides of $ZA\,Y+YZ$ other than YZ and such that $(RST+\text{Int } RST)$ lies, except for $RS+ST$, on Int $(ZA\,Y+YZ)$. If we show that Int $(ZA\,Y+YZ)\subset$Int P, the triangle RST shows that the shelling theorem is satisfied for P. Consider

$$I(P) = \text{Int}(ZA\,Y + YZ) + \text{Int}(ZB\,Y + YZ) + \text{open segment } (YZ).$$

If we show that Int $P=I(P)$, it will follow that Int$(ZA\,Y+YZ)\subset$Int P. Recall that this was the last link in establishing the shelling theorem for P. It follows from the Schoenflies theorem applied to P that the complement of P has only one bounded component. Hence, we show that $I(P)=$Int P by showing that $I(P)$ and the complement $E(P)$ of $P+I(P)$ are mutually separated.

Since $P+I(P)$ is closed, $E(P)$ contains no limit point of $I(P)$. We show that $I(P)$ and $E(P)$ are mutually separated by showing that no point of $I(P)$ is a limit point of $E(P)$. Points of the open segment (YZ) are the only ones in question. No point of this open segment is a limit point of $E(P)$ if Int$(ZA\,Y+YZ)$ contains points on one side of (YZ) and Int$(ZB\,Y+YZ)$ contains points on the other side. This will be the case if Int$(ZA\,Y + YZ)$ does not intersect Int$(ZB\,Y+YZ)$ since each of the open sets has YZ on its boundary. The following two exercises show why these two open sets do not intersect.

Exercise 44. Why does $ZB\,Y$ not intersect Int $(ZA\,Y+YZ)$? Why does $ZA\,Y$ not intersect Int $(ZB\,Y+YZ)$?

Exercise 45. Show that there is no point common to Int $(ZA\,Y+YZ)$ and Int $(ZB\,Y+YZ)$.

J. W. Alexander described the *horned sphere* illustrated in Figure 32. It is topologically equivalent to the surface of a ball but its exterior is not like the exterior of a ball. Recall that a set topologically equivalent to the surface of a ball is called a 2-sphere but we sometimes use the word sphere alone for brevity.

We might describe the horned sphere as follows. A long cylinder closed at both ends is folded until the ends are near each other and parallel. Then tubes are pushed out each end until they hook as shown. The process is continued by pushing out additional tubes, etc. The resulting set has the property that,

although it is topologically equivalent to the surface of a sphere, there is a circle in the exterior of the horned sphere that cannot be shrunk in the exterior of the horned sphere to a point.

Although a horned sphere is topologically equivalent to a sphere, the complement of a horned sphere is not like the complement of a sphere because each simple closed curve in the complement of a sphere can be shrunk to a point without touching the sphere. The simple closed curve J shown in Figure 32 cannot be shrunk to a point in the complement of the horned sphere.

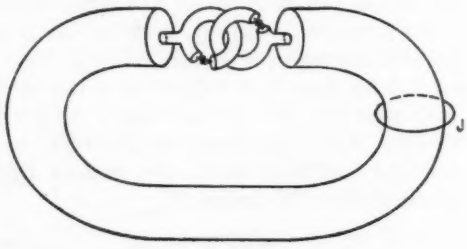


FIG. 32

The four-color problem. There are many unsolved problems in mathematics. It was not known for many years whether or not the Jordan curve theorem was true—though during part of this time many people suspected it to be true. The four color problem is still unsolved. It asks if each map on the surface of a sphere can be colored with four colors if the boundaries of the states are simple closed curves. States whose boundaries meet only at a point are allowed to have the same color, but states whose boundaries meet along an arc are not allowed to have the same color. If some of the states had boundaries that contained part of the graph of $y=\sin 1/x$, we might have a situation such as depicted in Figure 33 so it is essential to impose the condition that the boundaries of the states be simple closed curves.

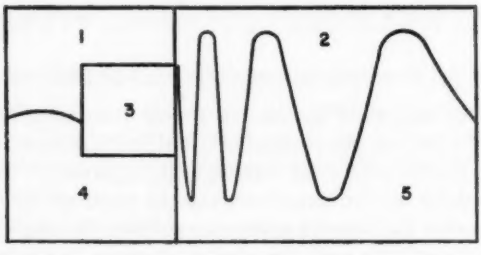


FIG. 33

8. PROPERTIES OF THE REAL NUMBERS

Much of mathematics is devoted to a study of the real numbers. In analytics much use is made of the fact that there is a 1-1 correspondence between the real numbers and the points on a line. Certain of the topological properties of a line can be developed from a study of the real numbers. With the view of getting a set of axioms that will characterize the line, we list certain properties of the set of real numbers E^1.

PROPERTY I. *E^1 is linearly ordered—that is, its elements a, b, c are ordered so that*:

1. *If $a \neq b$, either $a < b$ or $b < a$ (each pair of elements is related)*;
2. *$a \not< a$ (no element is less than itself)*;
3. *If $a < b$ and $b < c$, then $a < c$ (transitive property)*.

When we say that $a = b$, we mean that a is the same as b. Ordinarily we use the plural to denote more than one and, if we say that a, b are elements, we imply that they are distinct.

Any set (whether it is E^1 or not) is said to be *linearly ordered* if it has an order relation satisfying the above conditions.

Exercise 46. Show that if a, b are elements of a linearly ordered set, then not both $a < b$ and $b < a$.

An element a of a linearly ordered set X is said to be a *maximum* of X if for each other element b of X, $b < a$. Also, a is a *minimum* if for each other element b of X, $a < b$.

PROPERTY II. *E^1 has no maximum or minimum.*

Exercise 47. Give an example of a linearly ordered set failing to satisfy Property II.

If A, B are two (nonnull) subsets of a linearly ordered set X, then $[A, B]$ is called a *cut* of X if (1) each point of X belongs to either A or B, and (2) each point of A is less than each point of B.

Exercise 48. (a) Give an example of a cut $[A, B]$ of the reals E^1 so that A has no maximum. (b) Give an example of a cut $[D, E]$ of E^1 so that D has a maximum. (c) In which of the above cases does B or E have a minimum?

PROPERTY III. *E^1 satisfies the Dedekind cut condition—that is, if $[A, B]$ is a cut of E^1, either A has a maximum or B has a minimum.*

Exercise 49. Give an example of set X satisfying Properties I, II, but not III.

Exercise 50. Give an example of a set X different from E^1 satisfying Properties I, II, III.

A set is called *countable* if either it is finite or its elements can be put in a one-to-one correspondence with the positive integers—that is they can be arranged in a sequence. It may be shown that each subset of a countable space is countable.

A set is called *uncountable* if it is not countable. It is known that the rational numbers are countable while the real numbers are uncountable.

a is between b and c if either $b<a<c$ or $c<a<b$.

PROPERTY IV. *E^1 is separable—that is, it contains a countable set of points $p_1, p_2, \cdots$ such that if a, b are any two points of E^1 there is some p_i between a and b.*

Exercise 51. What is a countable set of numbers that shows the real numbers are separable?

Two elements of a linearly ordered set are *adjacent* if there is no element between them.

Exercise 52. Show that no two points of a set satisfying Property IV are adjacent.

If a, b are two points, the set consisting of a, b and all points between a and b is called a *closed interval*. If a, b are adjacent, the corresponding closed interval contains only two points.

Two closed intervals *intersect* if some point belongs to both.

Exercise 53. Show that if G is a collection of closed intervals of the real numbers and no two elements of G intersect, then G is countable.

Example 1. Consider the points of a horizontally based square linearly ordered as follows: $a<b$ means that either a, b are on the same vertical line and b is above a or a lies on a vertical line to the left of b.

Exercise 54. Show that above example does not satisfy Properties II, IV. How may it be modified so as to satisfy Property II?

Example 2. Suppose that each finite ordered set of letters is a *word*. Hence CAT, AAAC, ZZRNT, A, NAACP, KKK, etc., are words. Suppose the words are linearly ordered as would be done in an alphabetical ordering.

Exercise 55. Example 2 satisfies Property I. Show that it has the following properties: a. It has a minimum but no maximum. b. It does not satisfy Property III. c. There is no word preceding CAT which is adjacent to CAT but there is a word following CAT which is adjacent to CAT. d. There are only a countable number of words in the set.

Unsolved problem. Mathematicians have shown that any set L satisfying Properties I, II, III, IV is like the set of reals E^1 in the sense that there is a 1-1 correspondence between the elements of L and the numbers of E^1 that preserves the order relation. It is an unsolved problem as to whether or not any set satisfying Properties I, II, III, IV′, IV″ is like a line where IV′, IV″ are as follows:

PROPERTY IV′. *No two elements of L are adjacent.*

PROPERTY IV″. *Any collection of nonintersecting closed intervals of L is countable.*

This question was raised (in French) in a Polish mathematical journal in 1920 by a Russian mathematician named Souslin. It is called the Souslin problem. Despite many attempts to solve it, it remains unsolved today. One could solve it by either showing that Properties I, II, III, IV′, IV″ imply Property IV or by constructing an example satisfying Properties I, II, III, IV′, IV″ but not Property IV. Of course, one should not do both.

9. A TOPOLOGICAL CHARACTERIZATION OF A LINE

We use the properties of the real numbers considered in the preceding section to give a topological characterization of a line. This characterization is a set of axioms satisfied by the line such that anything that satisfies the axioms is topologically equivalent to a line.

A space is a linear topological space if its points are ordered and its neighborhoods are the sets of the following three sorts—(1) the set of points between two fixed points, (2) the set of points before some fixed point, or (3) the set of points following some fixed point. In studying a line, we only need the first kind of neighborhood but if we study a segment (which is also a linear topological space) the other types of neighborhoods are needed. (Sometimes other kinds of neighborhoods are needed but we do not find this convenient for our present purposes.)

A topological space is said to be *separable* if it contains a countable set of points such that each neighborhood contains one of these points.

The following is a topological characterization of a line L.

AXIOM 1. *L is a linear topological space.*

AXIOM 2. *L has no maximum or minimum.*

AXIOM 3. *L is connected.*

AXIOM 4. *L is separable.*

It may be shown that any space satisfying the above axioms is topologically equivalent to a line. Of course, there is no assurance that this space is "straight."

We recall some examples from the preceding section to see why all these axioms are necessary. To see that Axiom 2 is necessary (*i.e.*, Axioms 1, 3, 4 are not sufficient), consider a segment. To see that Axiom 3 is necessary, one could consider the set obtained by removing one point from a line. To see that Axiom 4 is needed, one could consider the points of the plane between and on two horizontal lines ordered as were the points in Example 1 of the preceding section.

Can Axiom 4 be replaced by the following?

AXIOM 4′. *L does not contain uncountably many mutually exclusive neighborhoods.*

There is no known proof that Axioms 1, 2, 3, 4′ characterize the line. Neither is there a known example of a space satisfying Axioms 1, 2, 3, 4′ that is not topologically equivalent to a line. This again is the Souslin problem.

Since there is 1-1 correspondence between points of a line and the real numbers that preserves order, the above axioms may be used to prove some useful theorems about real numbers. We illustrate with examples.

THEOREM. *A bounded collection of numbers has a least upper bound.*

We shall give a proof of this theorem based on the axioms. First we elaborate on its meaning.

A number x is an *upper bound* for a collection N of numbers if x is greater than or equal to each value of N. The *least upper bound* is the smallest of all the upper bounds. Similarly, a *lower bound* and a *greatest lower bound* are defined. A set of numbers with both an upper and a lower bound is called *bounded*.

Now for a proof of the theorem. Suppose N is a bounded collection of numbers. Let R be the set of all points p of the x-axis such that p is associated with an upper bound of N, and L be the rest of the x-axis. (See Fig. 34.)

FIG. 34

There is a point in R because N was given to be bounded. There is a point in L because any number less than a number in N is not an upper bound of N.

No point p of L is a limit point of R because if p is in L, there is a point q corresponding to a number of N such that q is to the right of p—no point to the left of q is a point of R.

Axiom 3 tells us that L, R cannot be mutually separated, so some point b of R is a limit point of L. This point will correspond to an upper bound x_b of N since b is in R. Since b is a limit point of L, for each positive number ϵ there is a point of N within ϵ of x_b. Then x_b is the least of all the upper bounds of N.

We now mention some theorems of a related nature frequently used in calculus.

1. *A collection of numbers with a lower bound has a greatest lower bound.*

2. *A bounded monotone increasing sequence of numbers converges to the least upper bound of the sequence.*

3. *The Dedekind cut proposition holds.*

(The proposition, sometimes taken as an alternative axiom, states that if $[A, B]$ is a cut of the reals, either A has a maximum or B has a minimum.)

4. *A bounded infinite collection of numbers has a greatest limit.*

(This limit is sometimes called the upper limit. If the numbers are regarded as points, it will be a limit point that is larger than any other limit point.)

Exercise 56. How would the x-axis be broken into two sets L, R to show that the above Theorem 4 holds?

Exercise 57. Show from the axioms that a segment is connected.

A subset of the plane is said to be bounded if it lies on the interior of some circle. It can be shown that each closed and bounded subset of the plane is bicompact and hence compact. Also, each compact subset of the plane is closed and bounded.

Exercise 58. Show that each compact subset of a line is closed and bounded.

Exercise 59. In showing that each closed and bounded subset of the x-axis is bicompact, one must show that if G is an open covering of a closed and bounded subset S of the x-axis, then a finite number of elements of G cover S. How would the x-axis be broken into two sets L, R to show that the assumption that no finite subcollection of G covers S leads to the contradiction that the x-axis is not connected?

The following theorems used in calculus and concerning a continuous function $f(x)$ defined on a segment $(a \leqq x \leqq b)$ are special cases of more general theorems we have considered.

1. *$f(x)$ is bounded.*

2. *$f(x)$ has a maximum and a minimum.*

3. *If $f(x)$ takes on two values, it takes on each value between them.*

The first of these results follows from the facts that a segment is bicompact (Exercise 59), its image under a continuous transformation is bicompact (Exercise 33) and hence compact (Exercise 34), and a compact set of numbers is closed and bounded (Exercise 58).

The second of these results follows from the further facts that a closed and bounded set of numbers has a least upper bound and a greatest lower bound since it is bounded. It takes on these values since it is closed.

The third result follows from the facts that a segment is connected (Exercise 57) and therefore its image is connected (Exercise 29).

A set X is said to have the *fixed-point property* if for each mapping f of X into itself, for some point p of X, $f(p) = p$. A circle does not have the fixed point property because a rotation can move each of its points. It can be shown that a segment, a disk, and a solid cube do have the fixed point property.

Exercise 60. Show that a segment has the fixed point property.

Unsolved problems. It is conjectured that if C is a bounded plane continuum whose complement is connected in the plane, then C has the fixed-point property. Partial solutions have been given of two types—one of these says that if the continuum has certain properties (such as being like a segment or a disk), it has the fixed-point property; the other type says that if certain restrictions are placed on the mapping, some point is left fixed. In spite of the vast amount of time spent on this problem by some talented workers, the answer still eludes us. Perhaps some student coming along will succeed in verifying the conjecture that has baffled so many of his predecessors—or maybe someone will construct a counterexample showing that the conjecture is false.

An arc is a set topologically equivalent to a segment. If one were asked to name a plane continuum with more than one point which was topologically equivalent to each continuum in it with more than a point, one might think of an arc (See Fig. 4). For many years no one knew of any example other than an arc, although the question was raised in mathematical journals as to whether or not there were any. In 1948 such an example was found. The problem as to whether or not there are still other examples is unsolved.

10. SOME EXAMPLES OF TOPOLOGICAL SPACES

What is a point?

If one is studying the real number system or a real line one might answer that a point is a real number. For analysis, the answer might be—a complex number. In plane analytics, a point is an ordered pair of real numbers—in solid analytics it is an ordered triple of real numbers.

In synthetic geometry and in topology, a point is an undefined term. If we have a collection of elements (which we call points), and certain subcollections of these elements (which we call neighborhoods), we may regard the original collections as a topological space if the neighborhoods satisfy certain conditions called axioms.

In this section we mention some abstract spaces somewhat out of the ordinary. We describe them by telling what the points and neighborhoods of these spaces are.

Euclidean 4-space. Here each point has four coordinates instead of three. (No mention need be made of time. In fact, any student who has worked problems with four variables may consider himself as having worked with 4-space.) A point is an ordered quadruple of numbers and a neighborhood with center at (x_0, y_0, z_0, w_0) and radius ϵ is the set of all points $(x, y, z\ w)$ such that $((x_0-x)^2+(y_0-y)^2+(z_0-z)^2+(w_0-w)^2)^{1/2}<\epsilon$.

Hilbert space. The points of this space are sequences $(a_1, a_2, \cdots)$ of numbers such that $\sum a_n^2$ exists. The distance between two points $(x_1, x_2, \cdots)$, $(y_1, y_2, \cdots)$ is $(\sum(x_n-y_n)^2)^{1/2}$. Neighborhoods are the interiors of generalized spheres.

A function space. A point in this abstract space is a continuous function $y=f(x)(0\leqq x\leqq 1)$. A neighborhood with center at $f_0(x)$ and radius ϵ is the collection of all functions $f(x)$ such that $|f(x)-f_0(x)|<\epsilon$ for each x between 0 and 1 inclusive.

Metric spaces. Many of the spaces we deal with in topology are metric spaces. *Metric spaces* are topological spaces with distance functions $\rho(p, q)$ (called the distance between p and q). *Distance functions* satisfy the following conditions.

1. $\rho(x, y)\geqq 0$, *the equality holding if and only if* $x=y$.
2. $\rho(x, y)=\rho(y, x)$ (*symmetry*).
3. $\rho(x, y)+\rho(y, z)\geqq\rho(x, z)$ (*triangle condition*).
4. $\rho(x, y)$ *preserves limit points.*

This fourth condition means that a point x belongs to the closure of a set M if and only if for each positive number ϵ, there is a point m of M such that $\rho(x, m) < \epsilon$.

Exercise 61. Describe a distance function for the above function space.

We shall now describe a topological space that has no distance function.

Another function space. This is a function space used extensively in studying linear spaces. Again a point in this abstract space is a function $y=f(x)$ $(0 \leqq x \leqq 1)$ but this time we do not suppose that it is continuous. To get a neighborhood in this function space, we select a finite collection of numbers $x_1, \cdots, x_n$ between 0 and 1 inclusive and a positive number ϵ. The neighborhood $N(f_0(x); x_1, \cdots, x_n; \epsilon)$ is the set of all functions $f(x)$ defined for $0 \leqq x \leqq 1$ and such that $|f_0(x_i) - f(x_i)| < \epsilon$ $(i=1, \cdots, n)$. To get another neighborhood, we select another function, another finite set of points, and another positive number.

If p is a limit point of a set X in a metric space, there is a sequence of distinct points of X converging to p.

In the function space we have just described, let $f_0(x)$ be the function such that $f_0(x) = 0$ for $0 \leqq x \leqq 1$. Let M be the collection of all functions that take on the value 0 at a finite collection of numbers and on the value 1 everywhere else. Although $f_0(x)$ is a limit point of M, no sequence of points of M converges to $f_0(x)$. This shows that the function space we have described is not a metric space.

Exercise 62. If $f_1(x), f_2(x), \cdots$, is a sequence of points of M in the above function space, what is a neighborhood of $f_0(x)$ that contains no point of $f_1(x), f_2(x), \cdots$?

Exercise 63. If p is a limit point of a set M in a metric space, how might one select a sequence of distinct points of M converging to p?

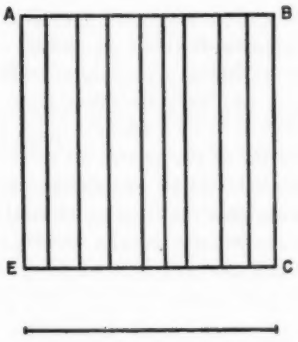


FIG. 35

Decomposition spaces. Suppose a square $ABCE$ plus its interior is decomposed into vertical straight line segments as shown in Figure 35. Let D be the

space whose " points" are these vertical segments and whose "neighborhoods" are the sum of such collections of " points" which are open subsets of the square disk. The decomposition space D is topologically equivalent to a segment.

Exercise 64. What "points" of D correspond to the ends of the segment equivalent to D?

If the square disk were decomposed in the manner illustrated in Figure 36, it is possible to define "points" and "neighborhoods" so that the decomposition space is topologically equivalent to the sum of two spherical surfaces and two segments.

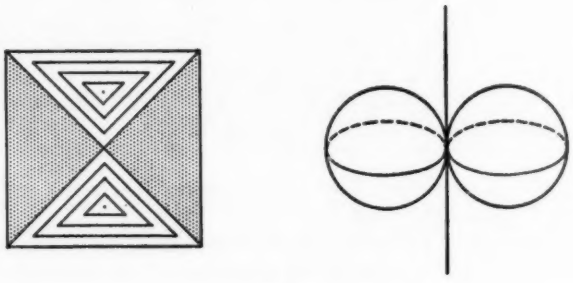


FIG. 36

Exercise 65. Which element of the decomposition space corresponds to the point of tangency of the two spheres? Which elements correspond to the outer ends of the two segments?

These two examples illustrate how topology may be used to study collections of figures rather than ordinary point sets.

The topologist is acquainted with dozens of different spaces—he may invent new spaces as the needs arise. Some of these spaces are much simpler than the ones we have described and others are more complex.

Queer spaces play an important role in topology. A wealth of examples of spaces with certain properties helps our understanding of these properties. It suggests to us that theorems are likely to follow from these properties.

Also, queer spaces are useful as counterexamples. There is no more convincing way of showing that one set of properties do not imply a second set than by exhibiting a space with the first set of properties but without the second set. Of the problems solved during recent years which had been regarded as difficult unsolved problems for many years, a goodly number of them were solved by the use of counterexamples.

11. SOME AXIOMS FOR TOPOLOGY

In geometry we may prove certain theorems on the basis of only part of our axioms—for example, we may omit the parallel axiom and study what theorems can be proved with the remaining set of axioms. While the axioms we then use do not completely describe the plane, we find that we can prove a restricted set of theorems on the basis of them. The beginning axioms we list are satisfied by a line, a plane, and E^3, but do not completely describe any of them. An advantage of working with such a restricted set of axioms is that the theorems we prove on the basis of these axioms will not only hold true on a line, in the plane, and in E^3, but in any space whatsoever that satisfies these axioms.

Suppose we have a collection S (which we call space) of things (which we call points) and certain subcollections of S (which we call neighborhoods). The points and neighborhoods may or may not be like the things we ordinarily think of as points and neighborhoods. We suppose however that they satisfy the following axioms.

AXIOM 1. *If a point p belongs to each of two neighborhoods N_1, N_2, then p lies in a neighborhood N_3 such that $N_3 \subset N_1 \cdot N_2$.*

AXIOM 2. *If p, q are two points, there are neighborhoods N_p, N_q containing p, q, respectively, such that $N_p \cdot N_q = 0$.*

We take "point" and "neighborhood" as undefined terms with the restriction that each neighborhood is a point set. Then without even knowing what these points and neighborhoods are, we can prove certain theorems on the basis of these axioms. Suppose that we have the neighborhood definition of limit point and the definitions of open, closed, sum, intersection, as given earlier. Then the following theorems can be proved.

THEOREM 1. *If p is a limit point of the sum of two sets, it is a limit point of one of them.*

THEOREM 2. *The intersection of two open sets is open.*

THEOREM 3. *The sum of two closed sets is closed.*

THEOREM 4. *A point set with only one point in it is closed.*

The spaces considered in the preceding section *Some examples of topological spaces* satisfied the two axioms we listed. These axioms are called the *Hausdorff axioms* and a space satisfying them is called a *Hausdorff space*. Hence, the four theorems are true in these spaces. Many additional theorems can be proved on the basis of them.

Exercise 66. An equivalent statement of Theorem 1 says that if p is not a limit point of either A or B, it is not a limit point of $A+B$. Prove this alternative statement. Which of the other three theorems is implied by Theorem 1?

Exercise 67. Theorem 2 can be proved from Theorem 3 by resorting to the results that the complements of closed sets are open and the complements of open sets are closed. However, prove it directly from Axiom 1.

Exercise 68. Which of the axioms is used to prove Theorem 4?

Exercise 69. Let $V(p, \epsilon)$ denote the set of all points whose distance from p is less than ϵ. To show that any metric space with the $V(p, \epsilon)$'s as neighborhoods satisfies Axiom 2, we might demonstrate that if p, q are two points, then $V(p, \rho(p, q)/2) \cdot V(q, \rho(p, q)/2) = 0$. How do we know that there is no point common to $V(p, \rho(p, q)/2)$ and $V(q, \rho(p, q)/2)$?

Exercise 70. To prove that a metric space satisfies Axiom 1 we might show that if r were a point in $V(p, \epsilon) \cdot V(q, \delta)$, then there is a positive number γ such that $V(r, \gamma) \subset V(p, \epsilon) \cdot V(q, \delta)$. What could we use for γ?

There are many theorems that could not be proved on the basis of Axioms 1 and 2 alone. For example, it could not be proved that a simple closed curve separates the plane. This result holds on the surface of a sphere or in a plane and is a consequence of the Jordan or Schoenflies theorems. However, additional axioms need to be assumed to prove these results from axioms. We frequently introduce additional axioms in studying topology. Such axioms as those given in the section *A topological characterization of a line* might be used. We might add an axiom saying that space is metric. We might axiomatize that neighborhoods are connected, that space is compact, that simple closed curves separate space, etc.

The following two exercises show why the space L of the section *A topological characterization of a line* is a Hausdorff space. If p and q are two points with $p < q$, we use (p, q) to denote the set of all points between p and q.

Exercise 71. To prove that L satisfies Axiom 1 of the present section, we show that if $r \epsilon (p_1, q_1) \cdot (p_2, q_2)$, then there is a (p_3, q_3) such that $r \epsilon (p_3, q_3) \subset (p_1, q_1) \cdot (p_2, q_2)$. What might we take for (p_3, q_3)?

Exercise 72. To show that L satisfies Axiom 2 of this section, we would consider two points p and q of L. Let r be a point between them, and let N_p, N_q be the sets of points preceding and following r respectively. Which of the axioms for L would not be satisfied if there were no point between p and q?

12. DEFINITIONS AND UNDEFINED TERMS

We continue our comparison of topology with geometry. We repeat some of the things we have said about definitions and undefined terms. First we consider undefined terms.

In giving definitions in an order, we find that the first terms to be considered do not have any predecessors. We take these as our undefined terms. In an intuitive sense, these undefined terms have a meaning for us. For example, in geometry we take the terms "set," "point," "line," "plane" as undefined terms. Although we may have an intuitive understanding of the meaning of these terms, we do not define them. If we develop our theorems logically without recourse to our intuitive notions about these terms, our results may have a wide interpretation.

We also have undefined terms in topology. We may have an intuitive feeling as to the meaning of an undefined term, though we may change this intuitive meaning without upsetting the logical structure of our study. In topology we take such words as "point," "neighborhood," "set" as undefined. We also suppose that we have a language at our command and know the meaning of such expressions as "is an element of." As a convenience we use previously understood terms from more elementary branches of mathematics.

Exercise 73. What represents a point in each of the following spaces? (a) The coordinate plane (Euclidean 2-space E^2). (b) Euclidean 3-space E^3. (c) Euclidean 4-space E^4. (d) Hilbert space. (e) The first function space given in the section *Some examples of topological spaces.* (f) The first decomposition space from that section.

In ordinary usage, a neighborhood of us is the set of things near us. There are neighborhoods of many sizes. In topology, our axioms tell us that a neighborhood is a point set but this is not a definition of a neighborhood since there are some point sets that are not neighborhoods. If we were restricting our attention to the plane, we would take a neighborhood to be the interior of a circle. In this case, our neighborhoods vary in size. We say that each point lies in an arbitrarily small neighborhood—meaning, that for each positive number ϵ, there is a neighborhood of the point of diameter less than ϵ. Instead of saying that a

set X is arbitrarily close to a point p, we say that each neighborhood containing p contains a point of X. Since we want our axioms to hold in a variety of spaces, we do not insist that neighborhoods be the interiors of circles. However, we use them much as if they were. Perhaps at the start it is convenient to think of a point as a geometric object and a neighborhood as the interior of a circle.

One of the things you may have gained from this article is an increased vocabulary so that you know the meaning of more of the terms of topology. However, definitions given in this article were not always given in the logical order. The four columns in Figure 37 not only give you an opportunity to recall certain of the definitions of topology but shows a possible logical order. The columns are arranged so that the terms in each column (after the first) may be defined by using terms from the preceding columns.

I	II	III	IV
Undefined terms	limit point	closed	
		closure	
		mutually separated	connected
Terms from other branches of mathematics		separate	
		compact	
(circle, ordered pair,	transformation	continuous transformation	reversibly continuous
1-1 correspondence,		inverse	fixed point property
interior of a circle,		onto	
numbers, is greater		uncountable	
than, etc.)	countable	separable	
	intersection	Hausdorff space	
Language of "every-day" vocabulary (contains, lies in, is an element of, etc.)	open set	open covering	bicompact
	complement		
	bounded plane set		
	subset		
	sum		
	difference		
	greatest lower bound		
	least upper bound		
	mutually exclusive		
	sequence converges		

FIG. 37

By adding another column we could include the term homeomorphism. The terms homogeneous and topologically equivalent would come in the next column. After we had defined topologically equivalent we could define arc, simple closed curve, and 2-sphere.

The reader may find it a good review of this article to refer to the list of terms following the next section and see how many of the definitions he can recall.

13. TRENDS TOWARD ABSTRACTNESS

One of the aspects of modern mathematics is the trend toward abstractness. This is not only true in topology but also in algebra, analysis, and other branches of mathematics. Let us examine how this trend has affected topology.

In proving a theorem in plane geometry, the student does not use all the properties of the plane but merely uses certain of the axioms. Similarly, in proving a topological theorem in the plane, one might note that his proof depends only on certain properties of the plane. The theorem would then hold true in any space with these essential properties. Let us illustrate this with an example.

THEOREM. *The sum of two closed plane sets is closed.*

This theorem can be proved without mentioning many properties of the plane. There is no need to mention the x-axis, y-axis, or straight lines. The theorem can be proved by using the definitions of closed sets, sum, limit point, along with the fact that if N_1, N_2 are two neighborhoods containing the same point, there is a neighborhood N_3 of the point in $N_1 \cdot N_2$ (Axiom 2 of the preceding section).

Exercise 74. Prove the above theorem on the basis of the definitions and the fact that if a point of the plane lies in each of two neighborhoods $N_1 \cdot N_2$, there is a neighborhood N_3 of the point in $N_1 \cdot N_2$.

As pointed out in the preceding section, we can prove the theorem about the sum of two closed sets being closed in any of the spaces mentioned in the section *Some examples of topological spaces* since all of these spaces satisfy Axiom 2 of the preceding section. One method of studying topology is to consider what theorems can be proved in a space with certain properties. One may not even know what space is under consideration as long as it is known that the space has the properties used in proving the theorem. Naturally the mathematician is

interested in getting the maximum in the way of results or conclusions from a minimum in the way of hypotheses and axioms.

If one is able to prove that certain theorems are true on the basis of certain properties of a space, at some time in the future a use may be found for a particular space with these properties, and the theorems proved before the space was invented will be known to hold.

An advantage (other than giving the imagination and mind ample working space) in dealing with abstract spaces is that our theorems tell us more than things about ordinary points and also reveal truths about collections of other sorts—for example, about collections of functions, such as described in Examples 3 and 4 in the section *Some examples of topological spaces*.

There are other abstract spaces which have for points other sorts of unexpected things—closed sets, continua, matrices.

Types of unsolved problems. We finish this article by mentioning some types of unsolved problems.

As we have indicated, some theorems go about as follows: Each space with a certain set of properties has some other property. Quite a few of the many mathematical papers published each year are devoted to proving this kind of theorem. We shall not discuss here the properties of interest but they include such things as being compact, bicompact, and having distance functions.

A more concrete type of theorem tells us that certain theorems (such as the Jordan curve theorem) hold in the plane or in some other familiar space. During the past forty years we have made much headway in proving theorems for the plane. In more recent years some headway has been made at hacking away at some of the complicated theorems in 3-space.

Intermediate between the abstract and concrete types of theorems are the characterization theorems. Show that a well-known space (such as a line, plane, or 3-space) satisfies a certain minimum set of conditions—then show that any space that satisfies these same conditions must be topologically equivalent to the well-known space.

We have some good topological characterizations of the line. However, one of the interesting unsolved problems (Souslin problem) is in this area. It gives a certain set of conditions satisfied by the line and asks if any space which satisfies them must be homeomorphic with a line.

Topological characterizations of the plane have been discovered during the past 40 years. An unsolved problem for many years in this area may be stated as follows: Is the connected compact metric space S topologically equivalent to the surface of a ball if it has more than one point and satisfies the following conditions?

1. *The neighborhoods of S are connected.*
2. *The complement of no simple closed curve is connected.*
3. *The complement of each pair of points is connected.*

The answer was given in the affirmative in 1946. The resulting characterization of a 2-sphere is called the Kline sphere characterization.

Although topological characterizations of 3-space have been given, these are still complicated. In view of the recent spurt toward increasing our knowledge of the topological properties of 3-space, there is hope for improvements here.

It is difficult to specify the types of unsolved problems that are of most interest to research topologists. Researchers vary in their interests and the field is wide. Things in which one worker is intensely interested may not even concern someone else at the moment, because he may be too busy working on something else. Although there are many unsolved problems in topology, those of interest are not likely to have easy solutions. However, we have talented young people coming along each year who are taking their places as ingenious researchers and making contributions that help push the frontier of mathematics farther ahead.

LIST OF TOPOLOGICAL TERMS

The following list showing where topological terms are defined in this article may be of convenience as a reference.

adjacent, 36
arc, 4, 41
band, 8
between, 36
bicompact, 27
bound—greatest lower, least upper, lower, upper, 34
boundary, 25
bounded in plane, 31
Cantor set, 26
closed, 19
closed interval, 36
closure, 19
compact, 27
common part, 17
complement, 14
component, 3, 26
connected, 3, 26
continuous, 4, 21, 22, 24
continuum, 26
converge, 25
countable, 36
cover, 27
cut, 36
decomposition spaces, 43
Dedekind cut condition, 36
difference, 18
distance function, 16, 42
domain, 18
domain of definition, 21
Euclidean spaces, 2, 42
fixed-point property, 40
four-color problem, 34
function spaces, 42
half-open segment, 5
Hausdorff axioms, 46
Hausdorff space, 46
Hilbert space, 42
homeomorphic, 5, 24
homeomorphism, 4, 24
homogeneous, 27
horned sphere, 33
image, 20
intersect, 36
intersection, 17
into, 21
inverse image, 20
Jordan curve theorem, 29
Klein bottle, 12
Kline sphere characterization, 51
limit point, 17
linear map, 22
linear topological space, 38
linearly ordered, 35
map, 22
mapping, 25
maximum, 35
metric space, 42
minimum, 35
Moebius band, 9
mutually exclusive, 26
mutually separated, 26
neighborhood, 16, 45
null set (empty set), 17
one-to-one transformation, 21
onto, 21
open covering, 27
open set, 18
open subset, 18
orientable, 13
point, 2, 45
point set, 17
product, 17
projective plane, 13
retraction theorem, 31
reversibly continuous, 24
Schoenflies theorem, 29, 31
segment, 5, 6
separable, 36, 38
separate, 3, 29
shelling theorem, 32
simple closed curve, 3, 28
Souslin problem, 37, 39
space, 2, 45
sphere, 10, 34
subset, 17
sum, 17
topologically equivalent, 3, 4, 5, 24
topology, 1, 27
torus, 11
transformation, 21
transitive property, 35
triangle condition, 43
uncountable, 36
union, 17

ANSWERS TO EXERCISES

1. Three. n.

2. Cube and ball, knotted curve and circle, theta curve and capital letter **B**, square disk pierced by segment and hemisphere with a tangent segment.

3. (A, R), (B), $(C, G, I, J, L, M, N, S, U, V, W, Z)$, (D, O), (E, F, T, Y), (H, K), (P, Q), (X). Yes—if the letters have width, each letter is equivalent to a disk or a disk with one or two holes in it.

4. The following pairs correspond: (c, z), (e, w), (f, v), (g, u). Either s or t might correspond to h. Any point on the circle $vxzy$ other than v and z might correspond to d.

5. A Moebius band.

6. The triangle with the upper left vertical edge and the one with the lower right vertical edge; or the triangle with the lower left vertical edge and the one with the upper right vertical edge; or the triangles with the leftmost and rightmost middle vertical sides.

7. A Moebius band.

8. A Moebius band. A projective plane.

9. Any function $f(x, y)$ satisfying the conditions $f(x, 0)=f(x, 1)$ and $f(0, y)=f(1, 1-y)$ will suffice. Examples of such functions are $f(x, y)=\cos 2\pi x \cos 2\pi y$ and $f(x, y)=x(x-1)(x-3)+y(y-1)(x+y)(x+y+1)^2(x+y-2)$.

10. A disk pierced by a segment and a hemisphere with a tangent segment, two homeomorphic bands with a different number of twists, some pair of the tori shown in Figure 13.

11. A Moebius band, a projective plane, a Klein bottle.

12. A Moebius band. We avoided mentioning a projective plane and a Klein bottle because they do not lie in E^3 and the sides of a surface refer to parts of the space in which they lie.

13. Any pair of the tori shown in Figure 13.

14. r and t are points of X; p, q, t are limit points.

15. p, q, s, t are limit points of X.

16. If $p \notin A$, $p \in B$ by the definition of $A+B$. Since B is open, p belongs to a neighborhood N in B by the definition of open. Then N lies in $A+B$.

17. Let r_p be the distance from p to the boundary of S. It is also permissible for it to be a fractional part of this distance.

18. (a) A line, (b) the interior of a square, (c) a segment without its end points, (d) the plane.

19. Show that each limit point of the closure belongs to the closure.

20. $\overline{G}$ is the sum of G, the segment from $(0, 1)$ to $(0, -1)$ and the set whose only element is the point $(1/\pi, 0)$.

21. Since p is a limit point of $A \cdot B$, each neighborhood of p contains a point of $A \cdot B - \{p\}$ by the definition of limit point. Then each neighborhood of p contains a point of $A - \{p\}$ since each point of $A \cdot B$ is a point of A. Then p is a limit point of A by the definition of limit point. Since A is closed, $p \in A$.

22. If each neighborhood of p contains a point of A, it contains a point of $A - \{p\}$ since $A = A - \{p\}$. Then p would be a limit point of A by the definition of limit point and $p \in A$ since A is closed.

23. If $p \notin A'$, $p \in A$ by the definition of A. Then p lies in a neighborhood N in A since A is open. No point of N lies in A' since each of its points lies in A.

24. Since $p \notin \bar{A}$, there is a neighborhood N of p that contains no point of A. Then N contains no point of $\bar{A}$ and p is not a limit point of $\bar{A}$.

25. (a) 1, 2, 7, 8. (b) 1, 5, 7. (c) 1, 2, 3, 4, 5, 8. (d) All.

26. (a) Projection of p on x-axis. (b) Projection of p on y-axis. (c) Distance from p to horizontal lines if they are equidistant from p. (d) Distance from p to vertical lines if they are equidistant from p.

27. If for each pair of horizontal planes, one above and one below p, there is a neighborhood of p such that each point of the graph in, above, or below the neighborhood is between the horizontal planes.

28. We show that each neighborhood U of $f(p)$ contains a point of $f(X)$. This insures that $f(p)$ is either a point or a limit point of $f(X)$. Let V be a neighborhood of p such that $f(V \cdot X) \subset U$. Since p is a limit point of X, V contains a point x of X and $f(x)$ is in U.

29. If $f(X)$ were the sum of two mutually separated (nonnull) sets A, B, then X would be the sum of the sets $f^{-1}(A)$, $f^{-1}(B)$. These two sets are mutually exclusive because f is single valued, and each is an open subset of X by Exercise 30. Hence if $f(X)$ were the sum of two mutually separated (nonnull) sets A, B, X would be the sum of two mutually separated (nonnull) sets $f^{-1}(A)$, $f^{-1}(B)$.

30. Suppose p is a point of $f^{-1}(D)$, where D is an open subset of Y. If U is a neighborhood of $f(p)$ with $U \cdot Y \subset D$, the continuity of f implies that there is a neighborhood V of p such that $f(V \cdot X) \subset U$. Then $V \cdot X \subset f^{-1}(D)$. The sum of all such V's for all points p of $f^{-1}(D)$ is an open set W and $W \cdot X = f^{-1}(D)$ is an open subset of X.

31. If U is an open subset of Y containing $f(p)$, we have that $f^{-1}(U)$ is an open subset of X, *i.e.*, $f^{-1}(U) = D \cdot X$, where D is open. We can use a neighborhood V of p in D to show that the definition of continuity is satisfied at p. Since f is continuous at each point p of X, it is continuous.

32. We show that each infinite collection $y_1, y_2, \cdots$ of points ($y_i \neq y_j$ if $i < j$) of $f(X)$ has a limit point in $f(X)$. Let x_i be a point of $f^{-1}(y_i)$. Then $x_1, x_2, \cdots$ has a limit point x_0 in X. Let K be infinite set formed by subtracting $f^{-1}f(x_0)$ from $x_0, x_1, x_2, \cdots$. Since x_0 is a limit point of K but $f(x_0)$ is not a point of $f(K)$ then, by Exercise 28, $f(x_0)$ is a limit point of $f(K)$ and hence of $y_1, y_2 \cdots$.

33. If G is a collection of domains covering $f(X)$ the inverse of each of these is an open subset of X by Exercise 30. If X is bicompact, a finite number of these inverses cover X and the collection of corresponding elements of G covers $f(X)$.

34. We show that if K is an infinite collection of points in a bicompact space X, then K has a limit point in X. For each point p of K let G_p be the sum of p and the complement of $\overline{K}$. No finite subcollection of the G_p's covers X so either the G_p's do not cover X or some G_p is not open. If some point q of X does not lie in any G_p, this point q is a limit point of K because it belongs to $\overline{K}$ but not to K. If some G_p is not open the corresponding point p is a limit point of K because if it were not, G_p would be the sum of two open sets—the complement of $\overline{K}$ and a neighborhood containing p but no other point of K.

35. We use functional notation to describe the homeomorphism—that is, the point corresponding to x is called $f(x)$. Let o be a point of the interior of the triangle and $f(o)$ be the center of the circle. For each point p on the triangle, let $f(p)$ be the point on the circle such that the ray from o through p points in the same direction as the ray from $f(o)$ through $f(p)$. Each point q of the first ray corresponds to the point $f(q)$ of the second ray such that $f(q)$ divides $f(o), f(p)$ in the same ratio (internally or externally) that q divides o, p. (See Fig. 27.)

36. First we describe a homeomorphism g which takes the plane onto itself and the circle with equation $x^2+y^2=1$ onto the given graph. If p is a point of the interior of the circle, $g(p)$ is on the same vertical line with p and divides the points where the vertical line hits the graph in the same ratio that p divides the points where the line hits the circle. If p is beneath the circle, $g(p)$ is on the same vertical line with p and the same distance below the graph as p is below the circle. If p is a point of the plane that is neither interior to, below, or on the lower semicircle of the circle, $g(p) = p$.

Suppose f is a homeomorphism such as given in Exercise 35 of the plane onto itself that takes the given triangle onto the circle with equation $x^2+y^2=1$. Then gf is the required homeomorphism —to find the point that corresponds to p, find the point $f(p)$ that corresponds to p under the homeomorphism f and then find the point $gf(p)$ which corresponds to this point $f(p)$ under g.

37. Construct segment AGC and segment BD as shown in Figure 28. Also construct the "kite-shaped" figure shown in Figure 29. We describe the homeomorphism f of the polygon $ABCDEF$ onto the triangle $A'C'D'$ by steps.

(i) Let f be a homeomorphism that takes AB linearly onto $A'B'$—if p is a point of AB, $f(p)$ divides A', B' in the same ratio that p divides A, B. Extend f to take BC, CD, DE, EF, FA, AG, GC, BD linearly onto $B'C'$, $C'D'$, $D'E'$, $E'F'$. $F'A'$, $A'G'$, $G'C'$, $B'D'$, respectively.

(ii) We now extend f so that it takes the interior of $ABCG$ onto the interior of $A'B'C'G'$. Let

p and $f(p)$ be points of the interiors respectively of these two triangles. If x is a point other than p of the interior of $ABCG$, extend the segment from p through x until it intersects triangle $ABCG$ at a point y. Now $f(y)$ has already been defined in the first step. The point $f(x)$ is the point that divides $f(p)$, $f(y)$ in the same ratio that x divides p, y.

(iii) Similarly extend f to take the interior of triangle BCD onto the interior of triangle $B'C'D'$.

(iv) To extend f to the interior of $ABDEF$, take a point p of this interior such that no ray from p intersects $ABDEF$ in two points (we might choose p on the line which bisects angle E) and let $f(p)$ be any point on the interior of $A'B'D'E'F'$. For each point x other than p of the interior of $ABDEF$, let y be the point where the ray from p through x intersects $ABDEF$ and $f(p)$ be the point that divides $f(p)$, $f(y)$ in the same ratio that x divides p, y.

(v) To extend the homeomorphism f to the exterior of $AGCDEF$, consider the same points p, $f(p)$ used in Step (iv). For each point x of the exterior of $AGCDEF$ let y be the point where the segment from p to x hits $AGCDEF$. Then $f(x)$ is the point that divides $f(p)$, $f(y)$ externally in the same ratio that x divides p, y.

38. So that $h(Y)$ lies on the segment from A to $h(X)$ and divides this segment in the same ratio that Y divides AX.

39. This follows by mathematical induction on $n(P)$ since $AB+AYB$ has fewer sides than P.

40. Let $UV=AB$.

41. Since X lies in Ext $(AB+AYB)$ (X is at a maximum distance from L) AXB lies except for its ends in this exterior. Then CE does not intersect AXB except possibly at C or E. Also, CE was chosen so as not to intersect AYB except at C and E since it intersects $AB+AYB$ only there.

42. $P-(CD+DE)+CE$.

43. gh.

44. B lies in Ext $(ZAY+YZ)$ as does each non-endpoint of ZBY. A lies in Ext $(ZBY+YZ)$ as does each non-endpoint of ZAY.

45. If there were such a point P, consider a segment PQ that misses YZ and has only the point Q on ZAY. We would have a contradiction of the first or second part of Exercise 44 according as PQ does or does not intersect ZBY.

46. If $a<b$ and $b<a$, then $a<a$ by the transitive property. This contradicts the fact that no element is less than itself.

47. The numbers between 0 and 1 inclusive.

48. (a) Let A be the set of negative numbers and B the others. (b) Let D be the set of non-positive numbers and E the others. (c) B has a minimum but E does not.

49. The set of rational numbers or the reals with 0 removed.

50. The integers. Another interesting example is the set of points in the plane on or between two horizontal lines ordered so that $p<q$ if p is below or to the left of q.

51. The rationals.

52. If some two points a, b were adjacent, there would be no point of the countable set $p_1, p_2, \cdots$ between them since there is no point whatsoever between them.

53. Let each of the intervals correspond to a rational number in it. This sets up a 1–1 correspondence between the intervals and a subset of the rationals. The set of rationals is countable so each subset of it is also. Hence, the collection of intervals is countable.

54. Example 1 does not satisfy Property II because the point of the square in the upper right hand corner is a maximum and the point in the lower left hand corner is a minimum. If these two points were removed, the remainder would satisfy Property II. Each vertical segment in the square is a closed interval in the linearly ordered set. The square contains uncountably many such vertical segments, no two of which intersect. Example 1 could not satisfy Property IV because it can be shown by techniques similar to those used in solving Exercise 53 that no space satisfying Property IV can have uncountably many mutually exclusive closed intervals.

55. a. Example 2 has the word A as a minimum. For each given word we can obtain a bigger word by adding another letter.

b. To check Property III, consider an expression such as ccc $\cdots$ (there are infinitely many

c's in this expression). It is not a word since only finite ordered sets of letters are words. If A is the set of words preceding ccc $\cdots$ and B is the set following, the cut $[A, B]$ does not satisfy the Dedekind cut condition.

c. The word CATA follows CAT and is adjacent to CAT. However, if a word precedes CAT, we obtain a new word between it and CAT by adding an A to the end of this original word.

d. There are 26 one-letter words, $(26)^2$ two-letter words, $(26)^3$ three-letter words, $\cdots$. We put these words in a 1-1 correspondence with the positive integers as follows: The one-letter words correspond to the integers between 1 and 26 inclusive, the two-letter words correspond to the integers between 27 and $26+(26)^2$ inclusively, the three-letter words correspond to the integers between $27+(26)^2$ and $26+(26)^2+(26)^3$ inclusively, $\cdots$.

56. Let L be the set of points of the x-axis associated with numbers n that have infinitely many numbers of the collection which are bigger than n while R is the rest of the x-axis.

57. Suppose pq is the segment with left end p and right end q, L_p is the part of the line to the left of p and L_q is the part of the line to the right of q. The segment pq is connected because the assumption that it is the sum of two mutually separated sets A, B leads to the contradiction that the line is not connected since it is the sum of two mutually separated sets A', B'; A' is A, $A+L_p$, $A+L_q$, $A+L_p+L_q$ according as A contains neither p nor q, A contains p but not q, A contains q but not p, A contains both p and q; B' is defined in a similar fashion.

58. Suppose that S is a compact (nonnull) subset of the line. Denote a fixed point of S by s_0. If S were not bounded, for each integer i there would be a point s_i in S which would not belong to the neighborhood with center at s_0 and radius i. This is impossible since S is compact but $s_1, s_2, \cdots$ would have no limit point. If S were not closed there would be a point x_0 not of S which is a limit point of S. For each integer i there would be a point s_i of S in the neighborhood with center at x_0 and radius $1/i$. This is impossible since S is compact and $s_1, s_2, \cdots$ would not have a limit point in S.

59. Let L be the collection of all points p of the x-axis such that a finite subcollection of G covers each point of S to the left of p. Let R be the rest of the x-axis.

60. Suppose f is a mapping of a horizontal segment S into itself, L is the set of points of S that move to the right under f, and R is the set of points of S that move to the left. Unless f leaves some point fixed, the left end of S belongs to L and the right end belongs to R. If a point p of S moves to the right, it follows from the continuity of f that there is a neighborhood of p such that each point of S in the neighborhood moves to the right. Hence, L is an open subset of S. Also, R is an open subset of S. Since S is connected (Exercise 56), it is not the sum of two mutually separated sets L and R. Any point of $S-(L+R)$ is left fixed under f.

61. $\rho(f(x), g(x))$ is the maximum value of $|f(x)-g(x)|$.

62. $N(f_0(x); y; \frac{1}{2})$ where y is a value such that $1=f_1(y)=f_2(y)=\cdots$.

63. For each integer i let m_i be a point of M whose distance from p is less than $1/i$ and such that m_i is different from any of $m_1, \cdots, m_{i-1}$. Then $m_1, m_2, \cdots$ converges to p.

64. The segments AE and BC.

65. The sum of the perimeter of the square and its diagonals. The centers of the upper and lower nested collections of triangles.

66. If p is not a limit point of A, it lies in a neighborhood N_1 which contains no point of $A-\{p\}$; also, if p is not a limit point of B it lies in a neighborhood N_2 which contains no point of $B-\{p\}$. It follows from Axiom 1 that p lies in a neighborhood $N_3 \subset N_1 \cdot N_2$. Then N_3 contains no point of $A+B-\{p\}$ and p is not a limit point of $A+B$. Theorem 3 is implied by Theorem 1.

67. Let p be a point in the intersection of two open sets U_1, U_2. We show that p lies in a neighborhood $N \subset U_1 \cdot U_2$. It follows from the definition of open sets that p lies in a neighborhood N_1 with $N_1 \subset U_1$ and p lies in a neighborhood N_2 with $N_2 \subset U_2$. It follows from Axiom 1 that p lies in a neighborhood $N_3 \subset N_1 \cdot N_2$. Then $N = N_3 \subset U_1 \cdot U_2$.

68. Axiom 2.

69. If r were a point common to $V(p, \rho(p, q)/2)$ and $V(q, \rho(p, q)/2)$, $\rho(p, r)+\rho(r, q)<\rho(p,q)$. This violates the triangle condition for a metric space.

70. Let γ be the minimum of $\epsilon-\rho(p, r)$ and $\delta-\rho(q, r)$.

71. Let p_2 be the larger of p_1, p_2 and q_3 be the smaller of q_1, q_2.

72. L would not be connected. Axiom 2.

73. (a) An ordered pair of numbers. (b) An ordered triple of numbers. (c) An ordered quadruple of numbers. (d) A sequence of numbers the sum of whose squares exists (is a number). (e) A continuous function $y=f(x)$ defined for $0 \leqq x \leqq 1$. (f) A vertical segment.

74. To show that the sum $A+B$ of two closed sets A, B is closed, we need to show that each limit point of $A+B$ belongs to $A+B$.

Suppose p is not a point of $A+B$. Then it is not a point of A (from the definition of sum). Since A is given to be closed, p is not a limit point of A (definition of closed). Hence, there is a neighborhood N_1 of p which contains no point of A (definition of limit point). Likewise, p is not a point or limit point of B and there is a neighborhood N_2 of p which contains no point of B.

There is a neighborhood N_3 of p which lies in both N_1 and N_2. (We cannot justify this from the definitions.) No point of A is in N_3 since no point of A is in N_2—also, no point of B is in N_3. Hence N_3 contains no point of $A+B$ since it contains no point of A and no point of $A+B$. Hence, $A+B$ is closed.

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